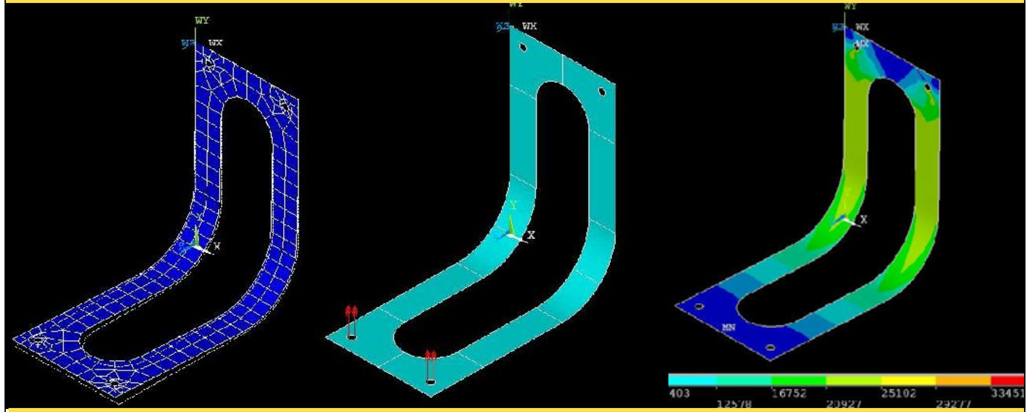
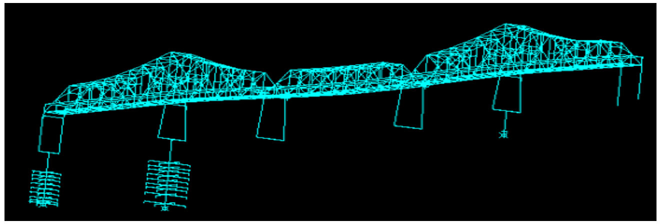
An Introduction to Finite Element Method



Engineering design



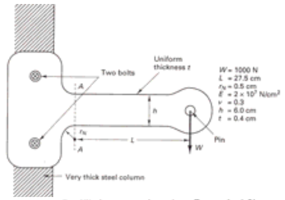
Physical Problem

Question regarding the problem
...how large are the deformations?
...how much is the heat transfer?

Mathematical model
Governed by differential equations

Assumptions regarding
Geometry
Kinematics
Material law
Loading
Boundary conditions
Etc.

Questions: What is the bending moment at section AA? What is the deflection at the pin?



Equilibrium equations (see Example 4.2)

$$\left.\begin{aligned}\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0\\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0\end{aligned}\right\} \text{in domain of bracket}$$

$\tau_{nn} = 0$, $\tau_{nt} = 0$ on surfaces except at point B and at imposed zero displacements

Stress-strain relation (see Table 4.3):

$$\begin{bmatrix}\tau_{xx}\\ \tau_{yy}\\ \tau_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \gamma_{xy}\end{bmatrix}$$

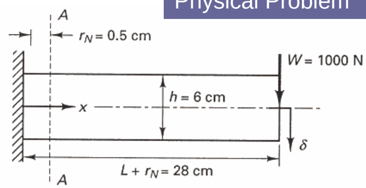
E = Young's modulus, ν = Poisson's ratio

Strain-displacement relations (see Section 4.2):

$$\epsilon_{xx} = \frac{\partial u}{\partial x};\quad \epsilon_{yy} = \frac{\partial v}{\partial y};\quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Exact

Physical Problem



Mathematical model

$M = WL$
$= 27{,}500$ N cm

$$\delta_{\text{at load W}} = \frac{1}{3}\frac{W(L+r_N)^3}{EI} + \frac{W(L+r_N)}{\frac{5}{6}AG}$$

$= 0.053$ cm

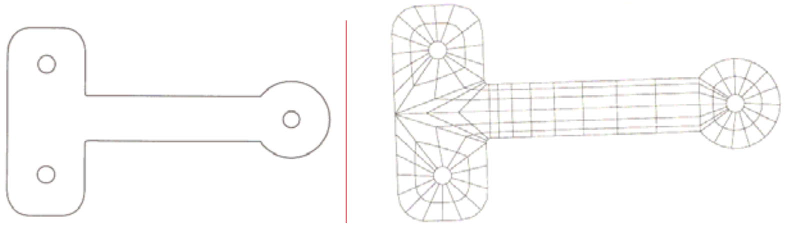
Difficult to solve by hand!

Finite element analysis

PREPROCESSING

1. Create a geometric model

2. Develop the finite element model (numerical approximation)

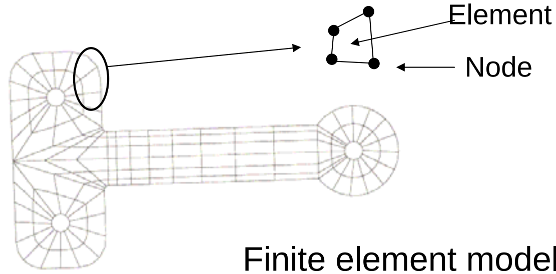


Solid model **Finite element model**

FEM analysis scheme

Finite element analysis

Step 1: Divide the problem domain into non overlapping regions ("**elements**") connected to each other through special points ("**nodes**")



Finite element model

Finite element analysis

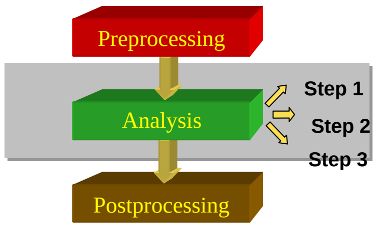
FEM analysis scheme

Step 2: Describe the approximate behavior of each element (spatially discretized by displacement-formulated FEM).

Step 3: Describe the approximate behavior of the entire body by putting together the behavior of each of the elements (this is a process known as "**assembly**")

POSTPROCESSING

Compute moment at section AA



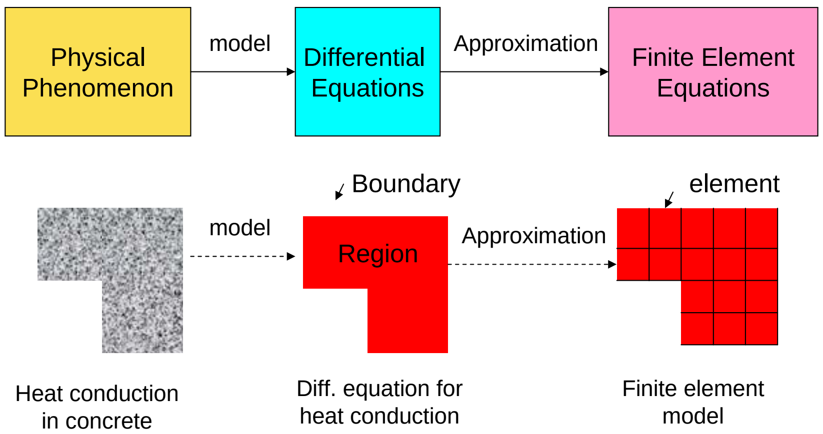
FEM solution to mathematical model 2 (plane stress)

Moment at section AA $M = 27{,}500\, N\, cm$

Deflection at load $\delta_{at\, load\, W} = 0.064\, cm$

Conclusion: **With respect to the questions we posed**, the beam model is ***reliable*** if the required bending moment is to be predicted within 1% and the deflection is to be predicted within 20%. The beam model is also highly ***effective*** since it can be solved easily (by hand).

Steps in engineeing mechanics analysis



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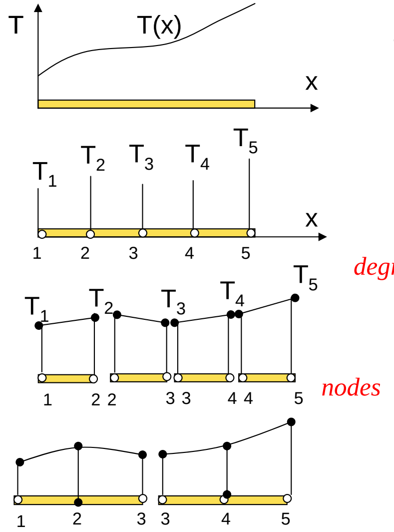
• Chapter 1 Overview



Temperature distribution along 1-dim. fin

Nodal points & temperature values at nodes

degrees of freedom

4 elements with linear temp within each element resulting an approximation along the fin.

nodes

2 elements with quadratic temp within each element resulting a better approximation

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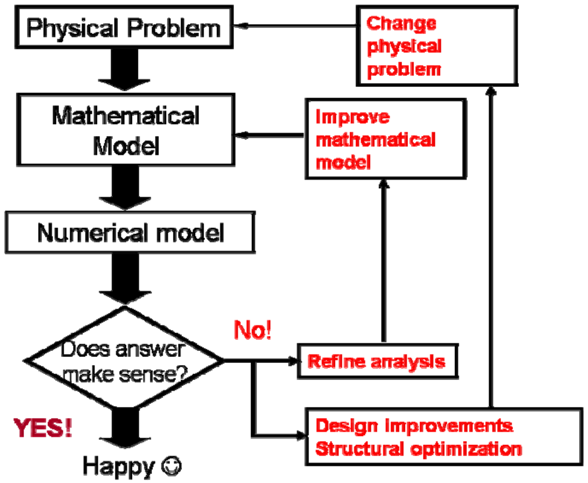

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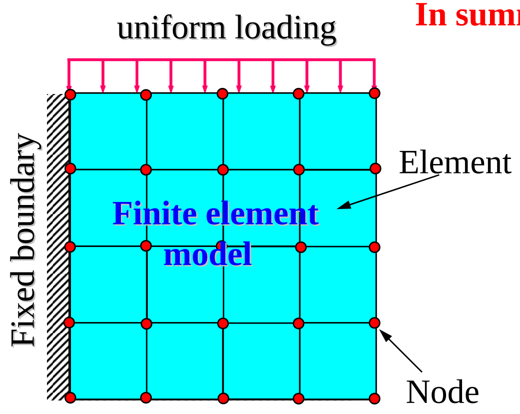
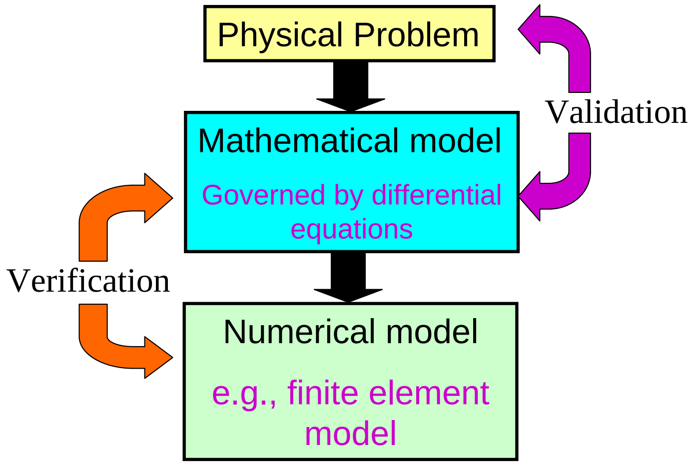
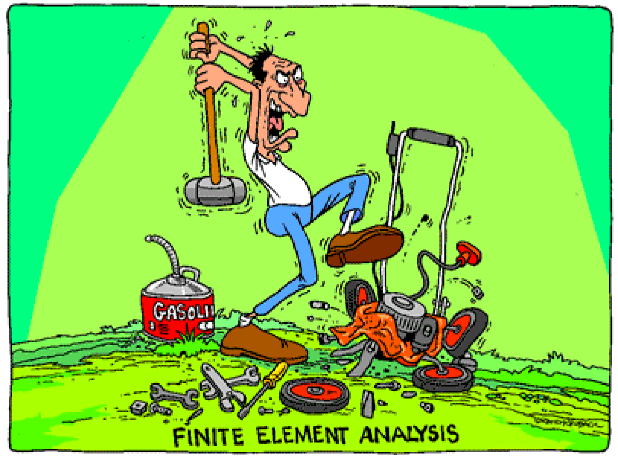


Problem: Obtain the stresses/strains in the plate

In summary, FEM involves…

- Approximate method
- Geometric model
- Node
- Element
- Mesh
- Discretization





Upcoming Course content

1. “Direct Stiffness” approach for springs
2. Bar elements and truss analysis
3. Introduction to boundary value problems: strong form, principle of minimum potential energy and principle of virtual work.
4. Displacement-based finite element formulation in 1D: formation of stiffness matrix and load vector, numerical integration.
5. Displacement-based finite element formulation in 2D: formation of stiffness matrix and load vector for CST and quadrilateral elements.
6. Discussion on issues in practical FEM modeling
7. Convergence of finite element results
8. Higher order elements
9. Isoparametric formulation
10. Numerical integration in 2D
11. Solution of linear algebraic equations

Summary:

- **Developing the finite element equations for a system of springs using the "direct stiffness" approach**
- **Application of boundary conditions**
- **Physical significance of the stiffness matrix**
- **Direct assembly of the global stiffness matrix**
- **Problems**

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• Direct Stiffness - springs

FEM analysis scheme

Step 1: Divide the problem domain into non overlapping regions ("**elements**") connected to each other through special points ("**nodes**")

Step 2: Describe the behavior of each element

Step 3: Describe the behavior of the entire body by putting together the behavior of each of the elements (this is a process known as "**assembly**")

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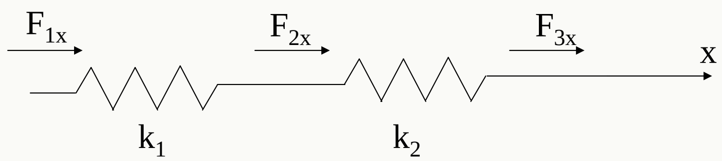

Problem

Analyze the behavior of the system composed of the two springs loaded by external forces as shown above

Given

F_{1x}, F_{2x}, F_{3x} are external loads. Positive directions of the forces are along the positive x-axis

k_1 and k_2 are the stiffnesses of the two springs

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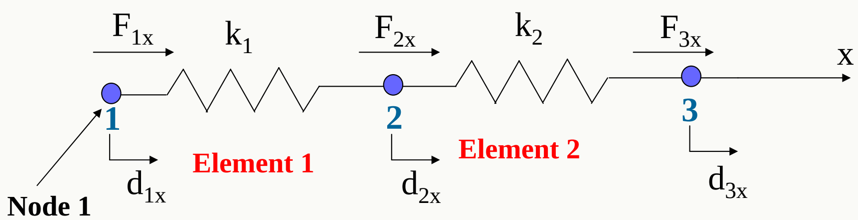
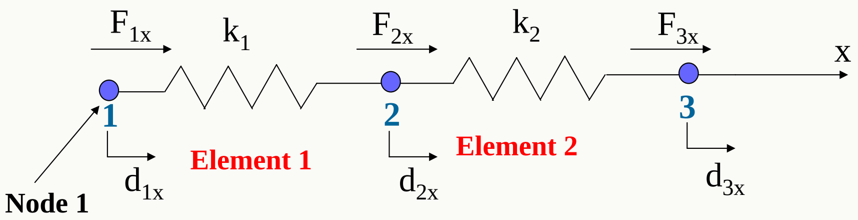
Solution

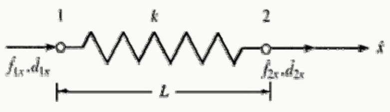
Step 1: In order to analyze the system we break it up into smaller parts, i.e., "elements" connected to each other through "nodes"



Unknowns: nodal displacements d_{1x}, d_{2x}, d_{3x},



Step 2: Analyze the behavior of a single element (spring)

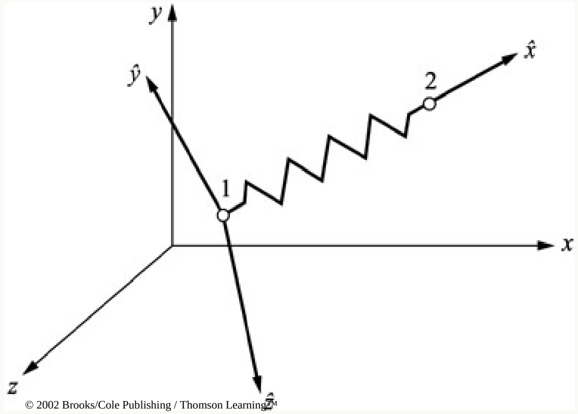


Two nodes: 1, 2

Nodal displacements: $\hat{d}_{1x}$ $\hat{d}_{2x}$

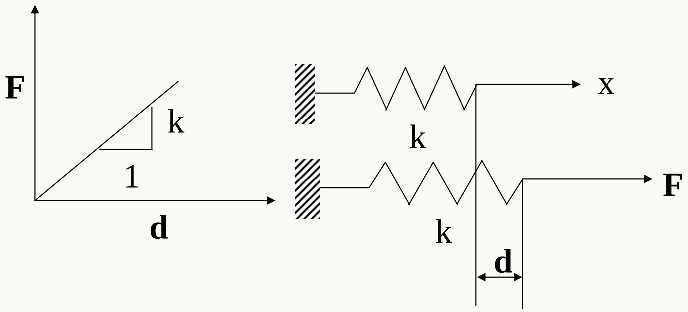
Nodal forces: $\hat{f}_{1x}$ $\hat{f}_{2x}$

Spring constant: k



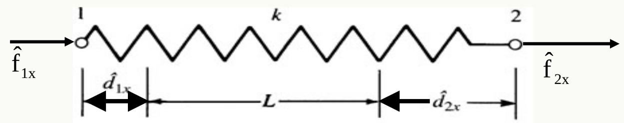
Local $(\hat{x},\hat{y},\hat{z})$ and global (x,y,z) coordinate systems

Behavior of a linear spring (recap)



Hooke's Law
F = kd

F = Force in the spring
d = deflection of the spring
k = "stiffness" of the spring



Hooke's law for our spring element $\hat{f}_{2x} = k(\hat{d}_{2x} - \hat{d}_{1x})$ Eq (1)

Force equilibrium for our spring element

$$\hat{f}_{1x} + \hat{f}_{2x} = 0$$

$$\Rightarrow \hat{f}_{1x} = -\hat{f}_{2x} = -k(\hat{d}_{2x} - \hat{d}_{1x})$$ Eq (2)

Collect Eq (1) and (2) in matrix form

$$\underline{\hat{f}} = \underline{\hat{k}}\,\underline{\hat{d}}$$

Element force vector; Element **stiffness matrix**; Element nodal displacement vector

$$\underbrace{\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix}}_{\hat{\underline{f}}} = \underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}_{\hat{\underline{k}}} \underbrace{\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}}_{\hat{\underline{d}}}$$

Note

1. The element stiffness matrix is "**symmetric**", i.e. $\hat{\underline{k}}^T = \hat{\underline{k}}$
2. The element stiffness matrix is **singular**, i.e.,

$$\det(\hat{\underline{k}}) = k^2 - k^2 = 0$$

The consequence is that the matrix is NOT invertible. It is not possible to invert it to obtain the displacements. Why?

The spring is not constrained in space and hence it can attain multiple positions in space for the same nodal forces

e.g.,

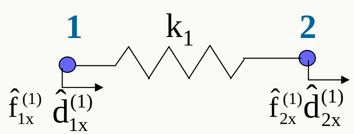
$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} 3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$$

Step 3: Now that we have been able to describe the behavior of each spring element, lets try to obtain the behavior of the original structure by assembly

Split the original structure into component elements

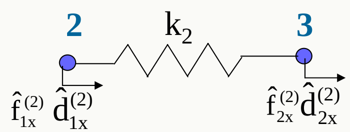
Element 1



$$\underbrace{\begin{Bmatrix} \hat{f}^{(1)}_{1x} \\ \hat{f}^{(1)}_{2x} \end{Bmatrix}}_{\hat{\underline{f}}^{(1)}} = \underbrace{\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}}_{\hat{\underline{k}}^{(1)}} \underbrace{\begin{Bmatrix} \hat{d}^{(1)}_{1x} \\ \hat{d}^{(1)}_{2x} \end{Bmatrix}}_{\hat{\underline{d}}^{(1)}}$$

Eq (3)

Element 2

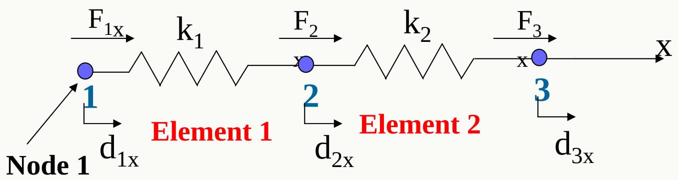


$$\underbrace{\begin{Bmatrix} \hat{f}^{(2)}_{1x} \\ \hat{f}^{(2)}_{2x} \end{Bmatrix}}_{\hat{\underline{f}}^{(2)}} = \underbrace{\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{\hat{\underline{k}}^{(2)}} \underbrace{\begin{Bmatrix} \hat{d}^{(2)}_{1x} \\ \hat{d}^{(2)}_{2x} \end{Bmatrix}}_{\hat{\underline{d}}^{(2)}}$$

Eq (4)

To assemble these two results into a single description of the response of the entire structure we need to link between the **local** and **global** variables.

Question 1: How do we relate the **local** (element) **displacements** back to the **global** (structure) displacements?



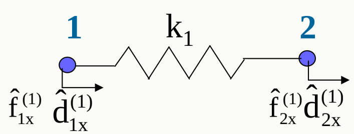
$$\hat{d}^{(1)}_{1x} = d_{1x}$$
$$\hat{d}^{(1)}_{2x} = \hat{d}^{(2)}_{1x} = d_{2x}$$
$$\hat{d}^{(2)}_{2x} = d_{3x}$$

Eq (5)

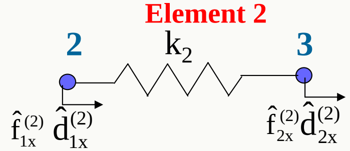
Hence, equations (3) and (4) may be rewritten as

$$\begin{Bmatrix} \hat{f}^{(1)}_{1x} \\ \hat{f}^{(1)}_{2x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix} \qquad \begin{Bmatrix} \hat{f}^{(2)}_{1x} \\ \hat{f}^{(2)}_{2x} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}$$

Element 1



Element 2



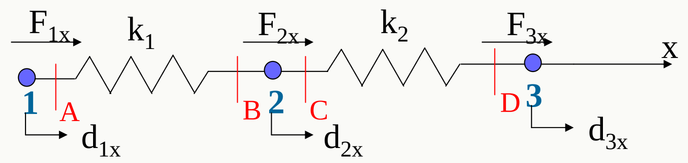
Or, we may **expand** the matrices and vectors to obtain

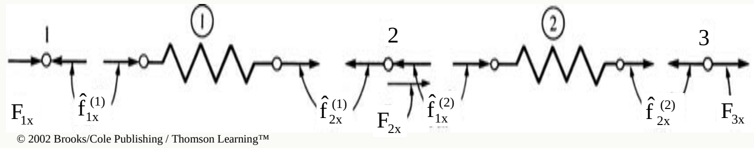
$$\underbrace{\begin{Bmatrix}\hat{f}^{(1)}_{1x}\\ \hat{f}^{(1)}_{2x}\\ 0\end{Bmatrix}}_{\underline{\hat{f}}^{(1)e}} = \underbrace{\begin{bmatrix} k_1 & -k_1 & 0\\ -k_1 & k_1 & 0\\ 0 & 0 & 0\end{bmatrix}}_{\underline{\hat{k}}^{(1)e}} \underbrace{\begin{Bmatrix} d_{1x}\\ d_{2x}\\ d_{3x}\end{Bmatrix}}_{\underline{d}} \qquad \textbf{Eq (6)}$$

$$\underbrace{\begin{Bmatrix}0\\ \hat{f}^{(2)}_{1x}\\ \hat{f}^{(2)}_{2x}\end{Bmatrix}}_{\underline{\hat{f}}^{(2)e}} = \underbrace{\begin{bmatrix} 0 & 0 & 0\\ 0 & k_2 & -k_2\\ 0 & -k_2 & k_2\end{bmatrix}}_{\underline{\hat{k}}^{(2)e}} \underbrace{\begin{Bmatrix} d_{1x}\\ d_{2x}\\ d_{3x}\end{Bmatrix}}_{\underline{d}} \qquad \textbf{Eq (7)}$$

$\underline{\hat{k}}^{(1)e}$ Expanded element stiffness matrix of element 1 (local)
$\underline{\hat{f}}^{(1)e}$ Expanded nodal force vector for element 1 (local)
$\underline{d}$ Nodal load vector for the entire structure (global)

Question 2: How do we relate the **local** (element) **nodal forces** back to the **global (**structure) forces? Draw 5 FBDs





$$\text{At node 1:}\quad F_{1x} - \hat{f}^{(1)}_{1x} = 0$$

$$\text{At node 2:}\quad F_{2x} - \hat{f}^{(1)}_{2x} - \hat{f}^{(2)}_{1x} = 0$$

$$\text{At node 3:}\quad F_{3x} - \hat{f}^{(2)}_{2x} = 0$$

In vector form, the nodal force vector (global)

$$\underline{F} = \begin{Bmatrix} F_{1x}\\ F_{2x}\\ F_{3x}\end{Bmatrix} = \begin{Bmatrix} \hat{f}^{(1)}_{1x}\\ \hat{f}^{(1)}_{2x} + \hat{f}^{(2)}_{1x}\\ \hat{f}^{(2)}_{2x}\end{Bmatrix}$$

Recall that the expanded element force vectors were

$$\underline{\hat{f}}^{(1)e} = \begin{Bmatrix}\hat{f}^{(1)}_{1x}\\ \hat{f}^{(1)}_{2x}\\ 0\end{Bmatrix} \quad \text{and} \quad \underline{\hat{f}}^{(2)e} = \begin{Bmatrix}0\\ \hat{f}^{(2)}_{1x}\\ \hat{f}^{(2)}_{2x}\end{Bmatrix}$$

Hence, the global force vector is simply the sum of the **expanded** element nodal force vectors

$$\underline{F} = \begin{Bmatrix} F_{1x}\\ F_{2x}\\ F_{3x}\end{Bmatrix} = \underline{\hat{f}}^{(1)e} + \underline{\hat{f}}^{(2)e}$$

But we know the expressions for the expanded local force vectors from Eqs (6) and (7)

$$\underline{\hat{f}}^{(1)e} = \underline{\hat{k}}^{(1)e}\underline{d} \quad \text{and} \quad \underline{\hat{f}}^{(2)e} = \underline{\hat{k}}^{(2)e}\underline{d}$$

Hence

$$\underline{F} = \underline{\hat{f}}^{(1)e} + \underline{\hat{f}}^{(2)e} = \underline{\hat{k}}^{(1)e}\underline{d} + \underline{\hat{k}}^{(2)e}\underline{d} = \left(\underline{\hat{k}}^{(1)e} + \underline{\hat{k}}^{(2)e}\right)\underline{d}$$

$$\boxed{\underline{F} = \underline{K}\,\underline{d}}$$

$\underline{F} =$ Global nodal force vector
$\underline{d} =$ Global nodal displacement vector
$\underline{K} =$ Global stiffness matrix
$=$ sum of expanded element stiffness matrices

For our original structure with two springs, the **global stiffness matrix** is

$$\underline{K}=\underbrace{\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\hat{\underline{k}}^{(1)e}}+\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}}_{\hat{\underline{k}}^{(2)e}}$$

$$=\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

NOTE

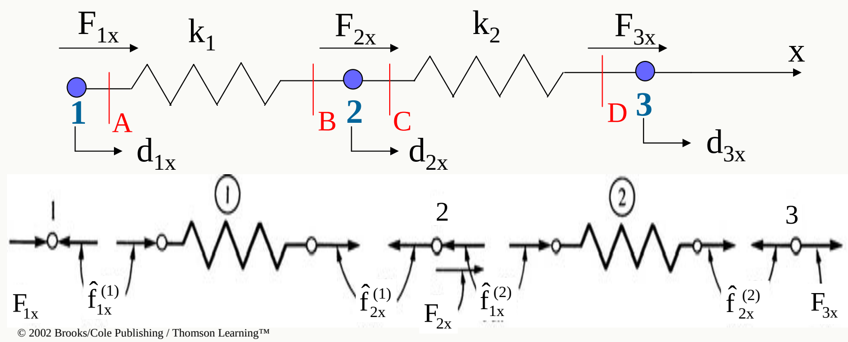
1. The global stiffness matrix is **symmetric**
2. The global stiffness matrix is **singular**

The system equations $\underline{F}=\underline{K}\,\underline{d}$ imply

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}=\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}$$

$$\Rightarrow \begin{aligned} F_{1x} &= k_1 d_{1x} - k_1 d_{2x} \\ F_{2x} &= -k_1 d_{1x} + (k_1+k_2) d_{2x} - k_2 d_{3x} \\ F_{3x} &= -k_2 d_{2x} + k_2 d_{3x} \end{aligned}$$

These are the 3 equilibrium equations at the 3 nodes.



At node 1: $F_{1x} - \hat{f}^{(1)}_{1x} = 0$

At node 2: $F_{2x} - \hat{f}^{(1)}_{2x} - \hat{f}^{(2)}_{1x} = 0$

At node 3: $F_{3x} - \hat{f}^{(2)}_{2x} = 0$

$$F_{1x} = k_1(d_{1x}-d_{2x}) = \hat{f}^{(1)}_{1x}$$

$$\begin{aligned} F_{2x} &= -k_1 d_{1x} + (k_1+k_2)d_{2x} - k_2 d_{3x} \\ &= -k_1(d_{1x}-d_{2x}) + k_2(d_{2x}-d_{3x}) \\ &= \hat{f}^{(1)}_{2x} + \hat{f}^{(2)}_{1x} \end{aligned}$$

$$F_{3x} = -k_2(d_{2x}-d_{3x}) = \hat{f}^{(2)}_{2x}$$

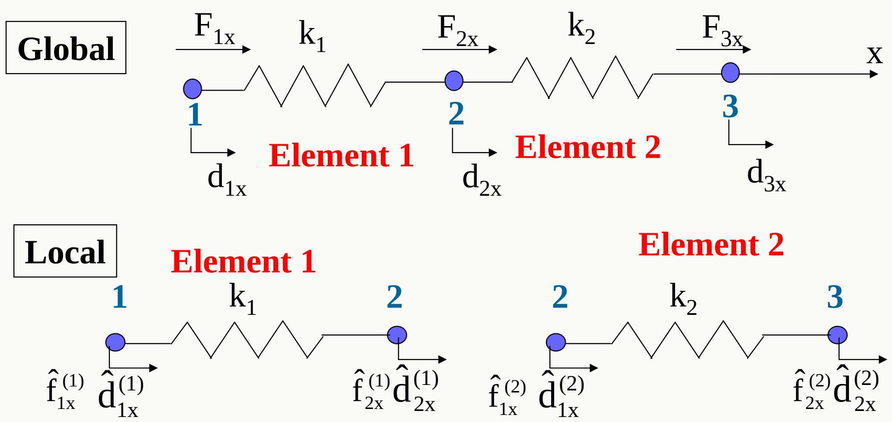
Notice that the sum of the forces equal zero, i.e., the structure is in static equilibrium.

$F_{1x} + F_{2x} + F_{3x} = 0$

Given the nodal forces, can we solve for the displacements?

To obtain unique values of the displacements, **at least one of the nodal displacements must be specified**.

Direct assembly of the global stiffness matrix



• Direct Stiffness - springs

Node element connectivity chart : Specifies the global node number corresponding to the local (element) node numbers

ELEMENT	Node 1	Node 2
1	1	2
2	2	3

Node 2 ← Local node number

2 ← Global node number

Stiffness matrix of element 1

$$\underline{\hat{k}}^{(1)} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{matrix} d_{1x} \\ d_{2x} \end{matrix}$$

(columns: d_{1x}, d_{2x})

Stiffness matrix of element 2

$$\underline{\hat{k}}^{(2)} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{matrix} d_{2x} \\ d_{3x} \end{matrix}$$

(columns: d_{2x}, d_{3x})

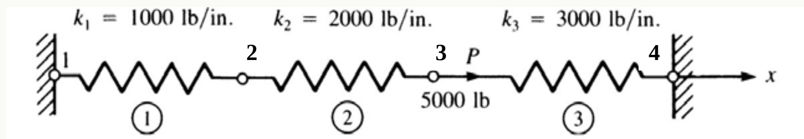
Global stiffness matrix

$$\underline{K} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{matrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{matrix}$$

(columns: d_{1x}, d_{2x}, d_{3x})

Examples:
Problems
2.1 & 2.3 of Logan

Example 2.1

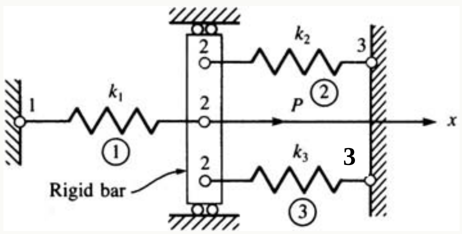


Compute the global stiffness matrix of the assemblage of springs shown above

$$\underline{K} = \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & (1000+2000) & -2000 & 0 \\ 0 & -2000 & (2000+3000) & -3000 \\ 0 & 0 & -3000 & 3000 \end{bmatrix} \begin{matrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{matrix}$$

(columns: d_{1x}, d_{2x}, d_{3x}, d_{4x})

Example 2.3



Compute the global stiffness matrix of the assemblage of springs shown above

$$\underline{K} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2+k_3 & -(k_2+k_3) \\ 0 & -(k_2+k_3) & (k_2+k_3) \end{bmatrix}$$

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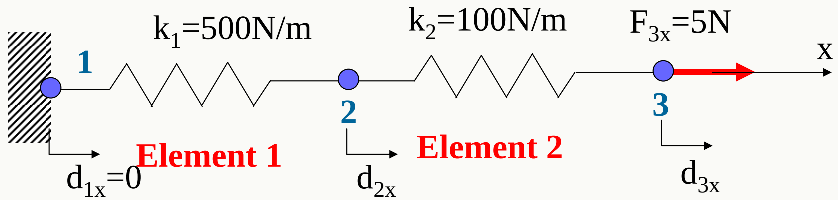
Imposition of boundary conditions

Consider 2 cases

Case 1: **Homogeneous** boundary conditions (e.g., $d_{1x}=0$)

Case 2: **Nonhomogeneous** boundary conditions (e.g., one of the nodal displacements is known to be different from zero)

Homogeneous boundary condition at node 1



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System equations

$$\begin{bmatrix} 500 & -500 & 0 \\ -500 & 600 & -100 \\ 0 & -100 & 100 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ 0 \\ 5 \end{Bmatrix}$$

(d_{1x} → 0)

Global Stiffness matrix — Nodal disp vector — Nodal load vector

Note that F_{1x} is the wall reaction which is to be computed as part of the solution and hence is an unknown in the above equation

Writing out the equations explicitly

$$-500d_{2x} = F_{1x} \qquad \text{Eq(1)}$$
$$600d_{2x} - 100d_{3x} = 0 \qquad \text{Eq(2)}$$
$$-100d_{2x} + 100d_{3x} = 5 \qquad \text{Eq(3)}$$

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Eq(2) and (3) are used to find d_{2x} and d_{3x} by solving

$$\begin{bmatrix} 600 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 0.01\,m \\ 0.06\,m \end{Bmatrix}$$

NOTICE: The matrix in the above equation may be obtained from the global stiffness matrix **by deleting the first row and column**

$$\begin{bmatrix} 500 & -500 & 0 \\ -500 & 600 & -100 \\ 0 & -100 & 100 \end{bmatrix} \Rightarrow \begin{bmatrix} 600 & -100 \\ -100 & 100 \end{bmatrix}$$

Note use Eq(1) to compute $F_{1x} = -500d_{2x} = -5N$

NOTICE:

1. Take care of **homogeneous** boundary conditions **by deleting the appropriate rows and columns** from the global stiffness matrix and solving the reduced set of equations for the unknown nodal displacements.

2. Both displacements and forces CANNOT be known at the same node. If the displacement at a node is known, the reaction force at that node is unknown (and vice versa)

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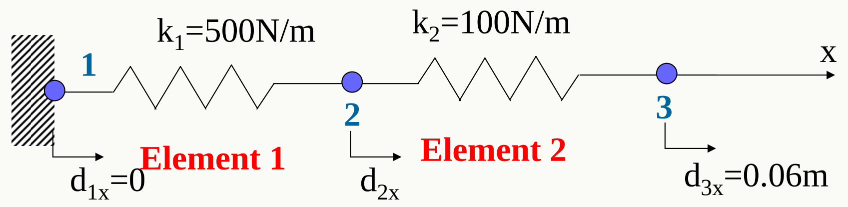
Imposition of boundary conditions…contd.

Nonhomogeneous boundary condition: spring 2 is pulled at node 3 by 0.06 m)



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• Direct Stiffness - springs

System equations

$$\begin{bmatrix} 500 & -500 & 0 \\ -500 & 600 & -100 \\ 0 & -100 & 100 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ 0 \\ F_{3x} \end{Bmatrix}$$

($d_{1x} \rightarrow 0$, $d_{3x} \rightarrow 0.06$)

Note that now F_{1x} and F_{3x} are not known.

Writing out the equations explicitly

$$-500 d_{2x} = F_{1x} \qquad \text{Eq(1)}$$

$$600 d_{2x} - 100(0.06) = 0 \qquad \text{Eq(2)}$$

$$-100 d_{2x} + 100(0.06) = F_{3x} \qquad \text{Eq(3)}$$

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• Direct Stiffness - springs

Now use only equation (2) to compute d_{2x}

$$600 d_{2x} = 100(0.06)$$

$$\Rightarrow d_{2x} = 0.01 m$$

Now use Eq(1) and (3) to compute F_{1x} =-5N and F_{3x}=5N

Recap of what we did

Step 1: Divide the problem domain into non overlapping regions (“**elements**”) connected to each other through special points (“**nodes**”)

Step 2: Describe the behavior of each element ($\underline{\hat{f}} = \underline{\hat{k}}\,\underline{\hat{d}}$) ← Element nodal displacement vector

Step 3: Describe the behavior of the entire body (by “**assembly**”).

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This consists of the following steps

1. Write the force-displacement relations of each spring in **expanded** form

$$\underline{\hat{f}}^{e} = \underline{\hat{k}}^{e}\,\underline{\hat{d}}$$

← Global nodal displacement vector

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Recap of what we did…contd.

2. Relate the local forces of each element to the global forces at the nodes (use FBDs and force equilibrium).

$$\underline{F} = \sum \underline{\hat{f}}^{e}$$

Finally obtain

$$\boxed{\underline{F} = \underline{K}\,\underline{d}}$$

Where the **global stiffness matrix**

$$\underline{K} = \sum \underline{k}^{e}$$

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Recap of what we did…contd.

Apply boundary conditions by partitioning the matrix and vectors

$$\begin{bmatrix} \underline{K}_{11} & \underline{K}_{12} \\ \underline{K}_{21} & \underline{K}_{22} \end{bmatrix} \begin{Bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{Bmatrix} = \begin{Bmatrix} \underline{F}_1 \\ \underline{F}_2 \end{Bmatrix}$$

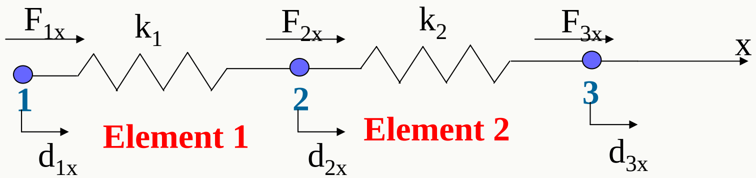
Solve for unknown nodal displacements

$$\underline{K}_{22}\underline{d}_2 = \underline{F}_2 - \underline{K}_{21}\underline{d}_1$$

Compute unknown nodal forces

$$\underline{F}_1 = \underline{K}_{11}\underline{d}_1 + \underline{K}_{12}\underline{d}_2$$

Physical significance of the stiffness matrix



In general, we will have a stiffness matrix of the form (assume for now that we do not know k_{11}, k_{12}, etc)

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

The finite element force-displacement relations:

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

The first equation is

$$k_{11}d_1 + k_{12}d_2 + k_{13}d_3 = F_1$$

Force equilibrium equation at node 1

Columns of the global stiffness matrix

What if $d_1=1$, $d_2=0$, $d_3=0$?

While nodes 2 and 3 are held fixed

$F_1 = k_{11}$ Force along node 1 due to unit displacement at node 1

$F_2 = k_{21}$ Force along node 2 due to unit displacement at node 1

$F_3 = k_{31}$ Force along node 3 due to unit displacement at node 1

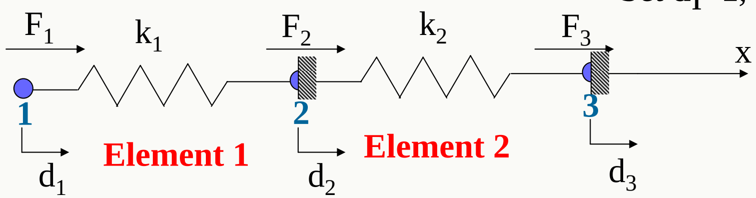
Similarly we obtain the physical significance of the other entries of the global stiffness matrix

In general

k_{ij} = Force at node 'i' due to **unit displacement** at node 'j' keeping **all the other nodes fixed**

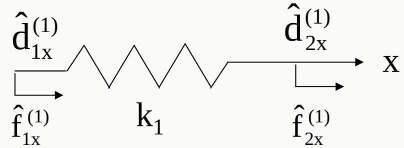
This is an alternate route to generating the global stiffness matrix e.g., to determine the **first column of the stiffness matrix**

Set $\mathbf{d_1=1}$, $d_2=0$, $d_3=0$



Find F_1=?, F_2=?, F_3=?

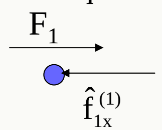
For this special case, Element #2 does not have any contribution. Look at the free body diagram of Element #1



$$\hat{f}_{2x}^{(1)} = k_1(\hat{d}_{2x}^{(1)} - \hat{d}_{1x}^{(1)}) = k_1(0-1) = -k_1$$

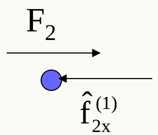
$$\hat{f}_{1x}^{(1)} = -\hat{f}_{2x}^{(1)} = k_1$$

Force equilibrium at node 1



$F_1 = \hat{f}_{1x}^{(1)} = k_1$

Force equilibrium at node 2



$F_2 = \hat{f}_{2x}^{(1)} = -k_1$

Of course, $F_3=0$

$F_1 = k_1 d_1 = k_1 = k_{11}$

$F_2 = -F_1 = -k_1 = k_{21}$

$F_3 = 0 = k_{31}$

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Hence the **first column** of the stiffness matrix is

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} k_1 \\ -k_1 \\ 0 \end{Bmatrix}$$

To obtain the **second column** of the stiffness matrix, calculate the nodal reactions at nodes 1, 2 and 3 when $d_1=0$, $\mathbf{d_2=1}$, $d_3=0$

Check that

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} -k_1 \\ k_1 + k_2 \\ -k_2 \end{Bmatrix}$$

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To obtain the **third column** of the stiffness matrix, calculate the nodal reactions at nodes 1, 2 and 3 when $d_1=0$, $d_2=0$, $\mathbf{d_3=1}$

Check that

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -k_2 \\ k_2 \end{Bmatrix}$$

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Steps in solving a problem

Step 1: Write down the **node-element connectivity table** linking local and global displacements

Step 2: Write down the **stiffness matrix of each element**

Step 3: **Assemble** the element stiffness matrices to form the global stiffness matrix for the entire structure using the node element connectivity table

Step 4: Incorporate appropriate **boundary conditions**

Step 5: Solve resulting set of **reduced equations** for the **unknown displacements**

Step 6: Compute the **unknown nodal forces**

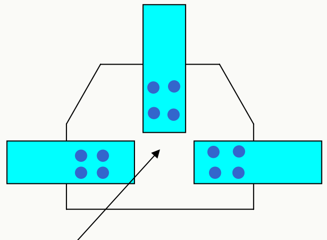
• Direct Stiffness – bar / truss

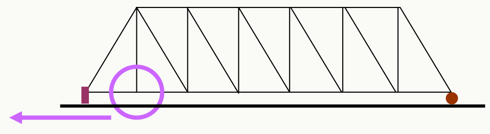
Summary:

- **Stiffness matrix of a bar/truss element**
- **Coordinate transformation**
- **Stiffness matrix of a truss element in 2D space**
- **Problems in 2D truss analysis (including multipoint constraints)**
- **3D Truss element**

Trusses: Engineering structures that are composed only of *two-force members. e.g., bridges, roof supports*

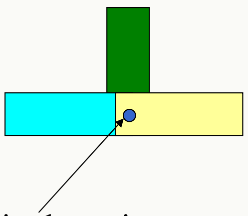
Actual trusses: Airy structures composed of slender members (I-beams, channels, angles, bars etc) joined together at their ends by welding, riveted connections or large bolts and pins

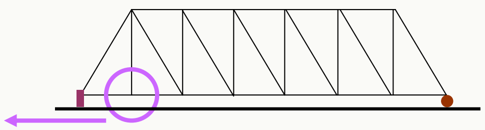

Gusset plate


A typical truss structure

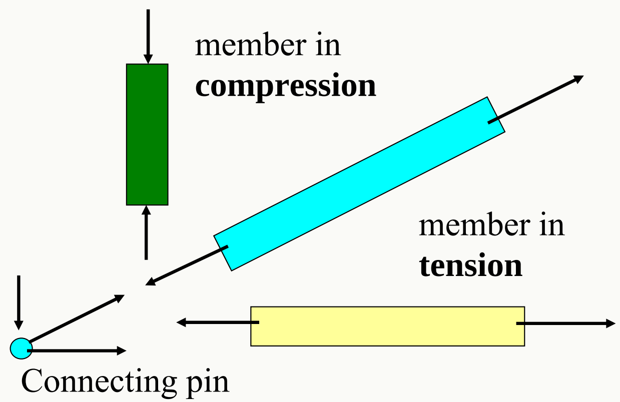
Ideal trusses: Assumptions

- Ideal truss members are connected only at their ends.
- Ideal truss members are connected by frictionless pins (no moments)
- The truss structure is loaded only at the pins
- Weights of the members are neglected


Frictionless pin


A typical truss structure

These assumptions allow us to idealize each truss member as a two-force member (members loaded **only** at their extremities by equal opposite and collinear forces)



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FEM analysis scheme

Step 1: Divide the truss into **bar/truss elements** connected to each other through special points ("**nodes**")

Step 2: Describe the behavior of each bar element (i.e. derive its **stiffness matrix** and **load vector** in local AND global coordinate system)

Step 3: Describe the behavior of the entire truss by putting together the behavior of each of the bar elements (by **assembling** their stiffness matrices and load vectors)

Step 4: Apply appropriate boundary conditions and solve

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Stiffness Matrix
A Bar/Truss Element

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Derivation: stiffness matrix of a bar element

L: Length of bar
A: Cross sectional area of bar
E: Elastic (Young's) modulus of bar
$\hat{u}(\hat{x})$:displacement of bar as a function of local coordinate $\hat{x}$ of bar
The **strain** in the bar at $\hat{x}$

$$\varepsilon(\hat{x}) = \frac{d\hat{u}}{d\hat{x}}$$

The **stress** in the bar (Hooke's law)

$$\sigma(\hat{x}) = E\,\varepsilon(\hat{x})$$

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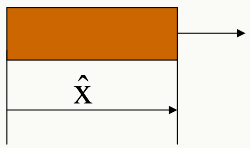
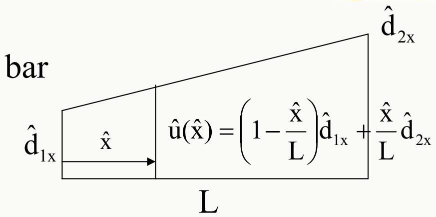


Tension in the bar
$T(\hat{x}) = EA\varepsilon$



Assume that the displacement $\hat{u}(\hat{x})$ is varying **linearly** along the bar

$$\hat{u}(\hat{x}) = \left(1 - \frac{\hat{x}}{L}\right)\hat{d}_{1x} + \frac{\hat{x}}{L}\hat{d}_{2x}$$

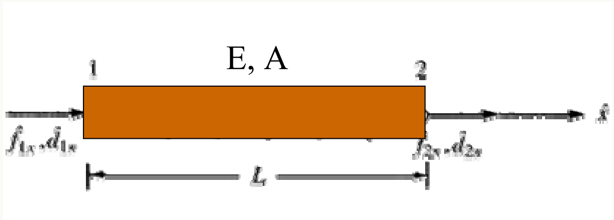
Then, strain is **constant** along the bar: $\varepsilon = \frac{d\hat{u}}{d\hat{x}} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$

Stress is also **constant** along the bar: $\sigma = E\varepsilon = \frac{E}{L}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$

Tension is **constant** along the bar: $T = EA\varepsilon = \underbrace{\frac{EA}{L}}_{k}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$

The bar is acting like a spring with stiffness $k = \frac{EA}{L}$

Recall the lecture on springs



Two nodes: 1, 2

Nodal displacements: $\hat{d}_{1x}$ $\hat{d}_{2x}$

Nodal forces: $\hat{f}_{1x}$ $\hat{f}_{2x}$

Spring constant: $k = \frac{EA}{L}$

Element stiffness matrix in local coordinates

$$\underline{\hat{f}} = \underline{\hat{k}}\,\underline{\hat{d}}$$

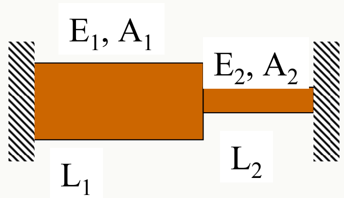
$$\underbrace{\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix}}_{\underline{\hat{f}}} = \underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}_{\underline{\hat{k}}} \underbrace{\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}}_{\underline{\hat{d}}}$$

Element force vector

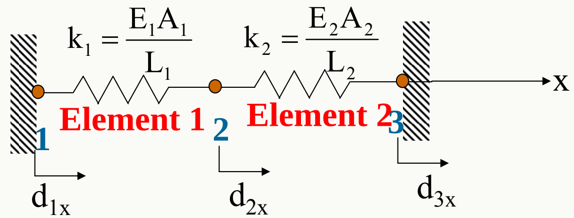
Element **stiffness matrix**

Element nodal displacement vector

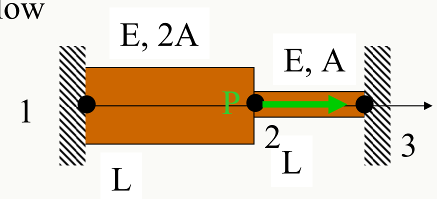
What if we have 2 bars?



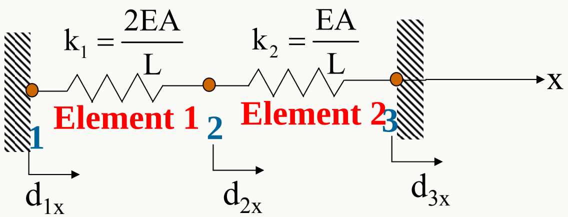
This is equivalent to the following system of springs



Problem 1: Find the stresses in the two-bar assembly loaded as shown below



Solution: This is equivalent to the following system of springs



We will first compute the displacement at node 2 and then the stresses within each element

---

The global set of equations can be generated using the technique developed in the lecture on "springs"

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

here $d_{1x} = d_{3x} = 0 \quad and \quad F_{2x} = P$

Hence, the above set of equations may be explicitly written as

$$-k_1 d_{2x} = F_{1x} \qquad (1)$$

$$(k_1 + k_2) d_{2x} = P \qquad (2)$$

$$-k_2 d_{2x} = F_{3x} \qquad (3)$$

From equation (2) $d_{2x} = \frac{P}{k_1 + k_2} = \frac{PL}{3EA}$

To calculate the **stresses**:

For element #1 first compute the element strain

$$\varepsilon^{(1)} = \frac{d_{2x} - d_{1x}}{L} = \frac{d_{2x}}{L} = \frac{P}{3EA}$$

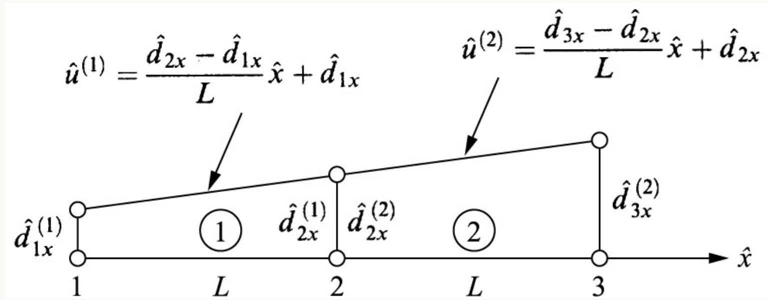
and then the stress as

$$\sigma^{(1)} = E\varepsilon^{(1)} = \frac{P}{3A} \qquad \text{(element in tension)}$$

Similarly, in element # 2

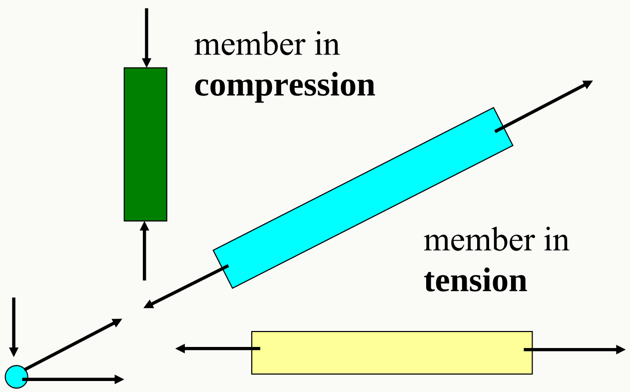
$$\varepsilon^{(2)} = \frac{d_{3x} - d_{2x}}{L} = -\frac{d_{2x}}{L} = -\frac{P}{3EA}$$

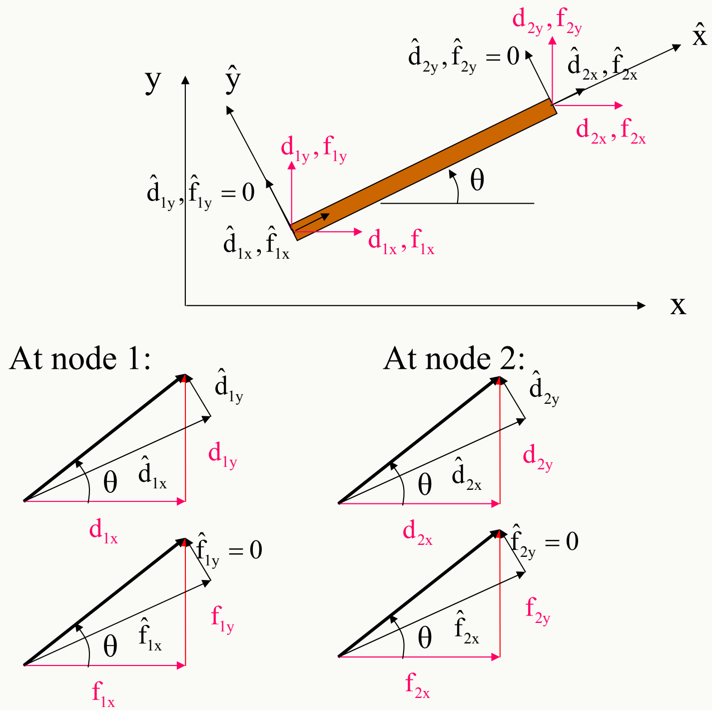
$$\sigma^{(2)} = E\varepsilon^{(2)} = -\frac{P}{3A} \qquad \text{(element in compression)}$$



Inter-element continuity of a two-bar structure

Bars in a truss have various orientations





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In the **global coordinate system**, the vector of nodal displacements and loads

$$\underline{d} = \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}; \qquad \underline{f} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}$$

Our objective is to obtain a relation of the form

$$\underset{4\times1}{\underline{f}} = \underset{4\times4}{\underline{k}} \ \underset{4\times1}{\underline{d}}$$

Where $\underline{k}$ is the 4x4 element stiffness matrix in global coordinate system

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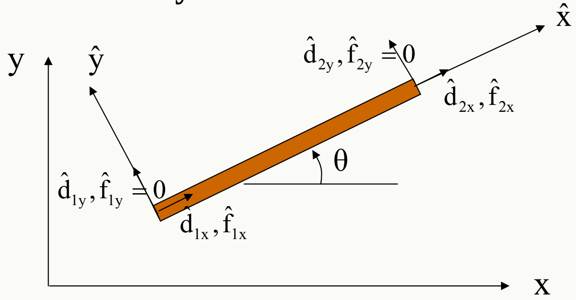
The key is to look at the local coordinates



$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$k = \frac{EA}{L}$$

Rewrite as

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix} = \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}$$

$$\boxed{\underline{\hat{f}} = \underline{\hat{k}} \ \underline{\hat{d}}}$$

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NOTES

1. **Assume** that there is **no stiffness** in the local $\hat{y}$ direction.

2. If you consider the displacement at a point along the local x direction as a vector, then the components of that vector along the global x and y directions are the global x and y displacements.

3. The expanded stiffness matrix in the local coordinates is symmetric and singular.

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NOTES

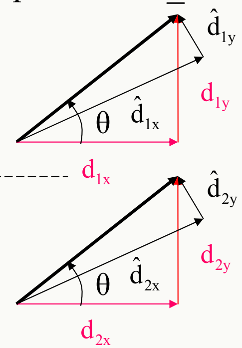
5. In local coordinates we have $\boxed{\underset{4\times1}{\underline{\hat{f}}} = \underset{4\times4}{\underline{\hat{k}}} \ \underset{4\times1}{\underline{\hat{d}}}}$

But or **goal** is to obtain the following relationship

$$\boxed{\underset{4\times1}{\underline{f}} = \underset{4\times4}{\underline{k}} \ \underset{4\times1}{\underline{d}}}$$

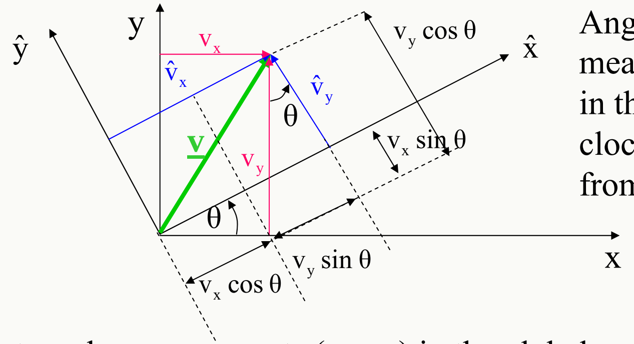
Hence, need a relationship between $\underline{\hat{d}}$ and $\underline{d}$ and between $\underline{\hat{f}}$ and $\underline{f}$

$$\underline{d} = \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix} \qquad \underline{\hat{d}} = \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}$$



Need to understand how the components of a vector change with coordinate transformation

Transformation of a vector in two dimensions



Angle θ is measured positive in the counter clockwise direction from the +x axis)

The vector $\underline{v}$ has components (v_x, v_y) in the global coordinate system and ($\hat{v}_x$, $\hat{v}_y$) in the local coordinate system. From geometry

$$\hat{v}_x = v_x \cos\theta + v_y \sin\theta$$

$$\hat{v}_y = -v_x \sin\theta + v_y \cos\theta$$

In matrix form

$$\begin{Bmatrix} \hat{v}_x \\ \hat{v}_y \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} v_x \\ v_y \end{Bmatrix}$$

Or

$$\begin{Bmatrix} \hat{v}_x \\ \hat{v}_y \end{Bmatrix} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{Bmatrix} v_x \\ v_y \end{Bmatrix}$$

where

Direction cosines

$$l = \cos\theta$$
$$m = \sin\theta$$

Transformation matrix for a single vector in 2D

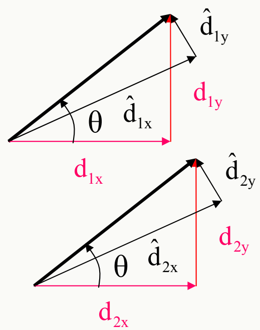
$$\underline{T}^* = \begin{bmatrix} l & m \\ -m & l \end{bmatrix}$$ relates $\underline{\hat{v}} = \underline{T}^* \underline{v}$

where $\underline{\hat{v}} = \begin{Bmatrix} \hat{v}_x \\ \hat{v}_y \end{Bmatrix}$ and $\underline{v} = \begin{Bmatrix} v_x \\ v_y \end{Bmatrix}$ are components of the **same vector** in local and global coordinates, respectively.

Relationship between $\underline{\hat{d}}$ and $\underline{d}$ for the truss element

At node 1 $\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \end{Bmatrix} = \underline{T}^* \begin{Bmatrix} d_{1x} \\ d_{1y} \end{Bmatrix}$

At node 2 $\begin{Bmatrix} \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix} = \underline{T}^* \begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix}$



Putting these together $\underline{\hat{d}} = \underline{T}\underline{d}$

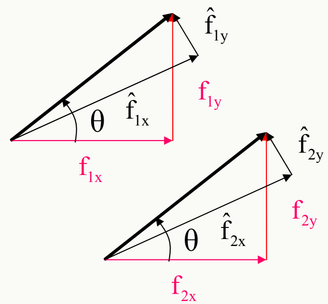
$$\underbrace{\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}}_{\underline{\hat{d}}} = \underbrace{\begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix}}_{\underline{T}} \underbrace{\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}}_{\underline{d}}$$

$$\underset{4\times4}{\underline{T}} = \begin{bmatrix} \underline{T}^* & \underline{0} \\ \underline{0} & \underline{T}^* \end{bmatrix}$$

Relationship between $\underline{\hat{f}}$ and $\underline{f}$ for the truss element

At node 1 $\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \end{Bmatrix} = \underline{T}^* \begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix}$

At node 2 $\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix} = \underline{T}^* \begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix}$



Putting these together $\underline{\hat{f}} = \underline{T}\underline{f}$

$$\underbrace{\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix}}_{\underline{\hat{f}}} = \underbrace{\begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix}}_{\underline{T}} \underbrace{\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}}_{\underline{f}}$$

$$\underset{4\times4}{\underline{T}} = \begin{bmatrix} \underline{T}^* & \underline{0} \\ \underline{0} & \underline{T}^* \end{bmatrix}$$

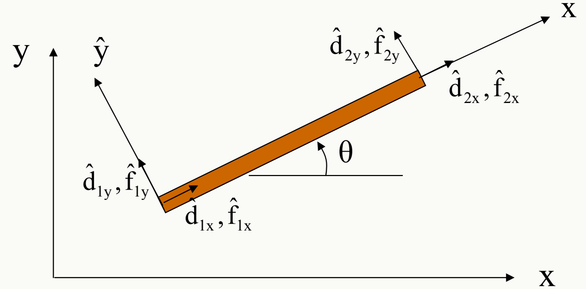
Important property of the transformation matrix $\underline{T}$

The transformation matrix is *orthogonal*, *i.e.* its inverse is its transpose

$$\boxed{\underline{T}^{-1} = \underline{T}^{T}}$$

Use the property that $l^2+m^2=1$

Putting all the pieces together



$$\boxed{\hat{\underline{f}} = \underline{T}\,\underline{f}}$$

$$\boxed{\hat{\underline{d}} = \underline{T}\,\underline{d}}$$

$$\hat{\underline{f}} = \hat{\underline{k}}\,\hat{\underline{d}}$$

$$\Rightarrow \underline{T}\underline{f} = \hat{\underline{k}}\,\underline{T}\underline{d}$$

$$\Rightarrow \underline{f} = \underbrace{\left(\underline{T}^{-1}\hat{\underline{k}}\,\underline{T}\right)}_{\underline{k}}\underline{d}$$

The desired relationship is $\boxed{\underset{4\times1}{\underline{f}} = \underset{4\times4}{\underline{k}}\ \underset{4\times1}{\underline{d}}}$

Where $\boxed{\underset{4\times4}{\underline{k}} = \underset{4\times4}{\underline{T}^{T}}\ \underset{4\times4}{\hat{\underline{k}}}\ \underset{4\times4}{\underline{T}}}$ is the **element stiffness matrix in the global coordinate system**

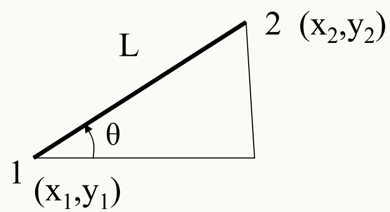
$$\underline{T} = \begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix} \qquad \hat{\underline{k}} = \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{k} = \underline{T}^{T}\hat{\underline{k}}\underline{T} = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Computation of the direction cosines

$$l = \cos\theta = \frac{x_2 - x_1}{L}$$

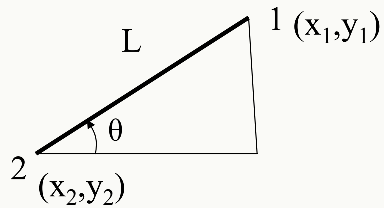
$$m = \sin\theta = \frac{y_2 - y_1}{L}$$



What happens if I reverse the node numbers?

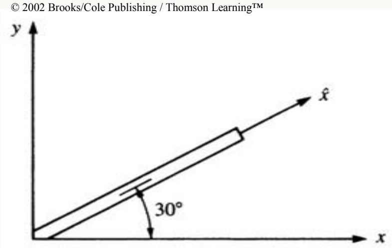
$$l' = \cos\theta = \frac{x_1 - x_2}{L} = -l$$

$$m' = \sin\theta = \frac{y_1 - y_2}{L} = -m$$



Question: Does the stiffness matrix change?

Example Bar element for stiffness matrix evaluation

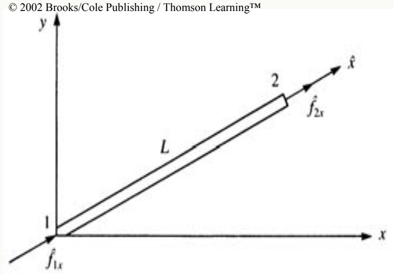


$$E = 30\times10^{6}\ psi$$
$$A = 2\ in^{2}$$
$$L = 60\ \ in$$
$$\theta = 30^{\circ}$$

$$l = \cos 30 = \frac{\sqrt{3}}{2}$$
$$m = \sin 30 = \frac{1}{2}$$

$$\underline{k} = \frac{(30\times10^{6})(2)}{60}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}\frac{lb}{in}$$

Computation of element strains



Recall that the element strain is

$$\varepsilon = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} = \frac{1}{L}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{1y}\\ \hat{d}_{2x}\\ \hat{d}_{2y}\end{Bmatrix}$$

$$= \frac{1}{L}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\underline{\hat{d}}$$

$$= \frac{1}{L}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\underline{T}\underline{d}$$

$$\varepsilon = \frac{1}{L}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix}\underline{d}$$

$$= \frac{1}{L}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\underline{d}$$

$$= \frac{1}{L}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\begin{Bmatrix}d_{1x}\\ d_{1y}\\ d_{2x}\\ d_{2y}\end{Bmatrix}$$

Computation of element stresses stress and tension

Recall that the element **stress** is

$$\sigma = E\varepsilon = \frac{E}{L}\left(\hat{d}_{2x} - \hat{d}_{1x}\right) = \frac{E}{L}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\underline{d}$$

Recall that the element **tension** is

$$T = EA\varepsilon = \frac{EA}{L}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\underline{d}$$

Steps in solving a problem

Step 1: Write down the **node-element connectivity table** linking local and global nodes; also form the **table of direction cosines** (l, m)

Step 2: Write down the **stiffness matrix of each element in global coordinate system with global numbering**

Step 3: **Assemble** the element stiffness matrices to form the global stiffness matrix for the entire structure using the node element connectivity table

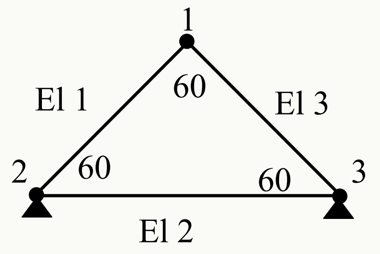
Step 4: Incorporate appropriate **boundary conditions**

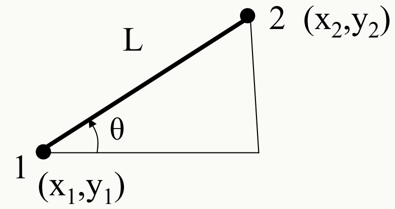
Step 5: Solve resulting set of reduced equations for the unknown displacements

Step 6: Compute the unknown nodal forces

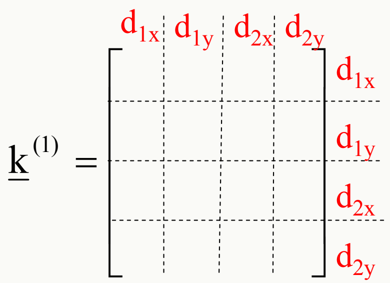
Node element connectivity table

ELEMENT	Node 1	Node 2
1	1	2
2	2	3
3	3	1

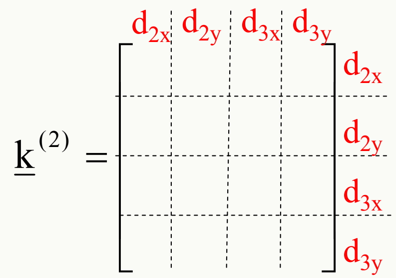




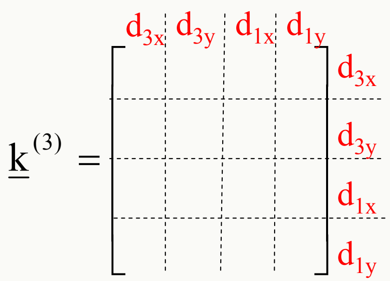
Stiffness matrix of element 1



Stiffness matrix of element 2

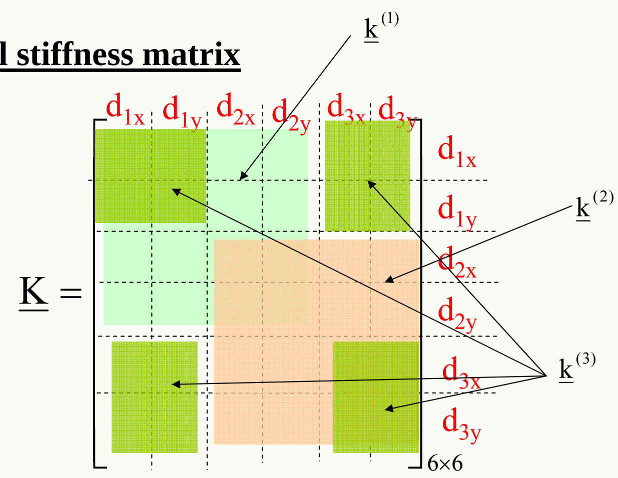


Stiffness matrix of element 3



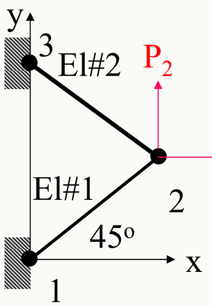
There are 4 **degrees of freedom** (**dof**) per element (2 per node)

Global stiffness matrix



How do you incorporate **boundary conditions**?

Example 2



The length of bars 12 and 23 are equal (L)
E: Young's modulus
A: Cross sectional area of each bar
Solve for
(1) d_{2x} and d_{2y}
(2) Stresses in each bar

Solution

Step 1: Node element connectivity table

ELEMENT	Node 1	Node 2
1	1	2
2	2	3

Table of nodal coordinates

Node	x	y
1	0	0
2	Lcos45	Lsin45
3	0	2Lsin45

Table of direction cosines

ELEMENT	Length	$l = \frac{x_2 - x_1}{length}$	$m = \frac{y_2 - y_1}{length}$
1	L	cos45	sin45
2	L	-cos45	sin45

Step 2: Stiffness matrix of each element in global coordinates with global numbering

Stiffness matrix of element 1

$$\underline{k}^{(1)} = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Columns: d_{1x}, d_{1y}, d_{2x}, d_{2y}; rows: d_{1x}, d_{1y}, d_{2x}, d_{2y}

$$= \frac{EA}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

Stiffness matrix of element 2

Columns: d_{2x}, d_{2y}, d_{3x}, d_{3y}; rows: d_{2x}, d_{2y}, d_{3x}, d_{3y}

$$\underline{k}^{(2)} = \frac{EA}{2L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Step 3: Assemble the global stiffness matrix

$$\underline{K} = \frac{EA}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

The final set of equations is $\underline{K}\,\underline{d} = \underline{F}$

Step 4: Incorporate boundary conditions

$$\underline{d} = \begin{Bmatrix} 0 \\ 0 \\ d_{2x} \\ d_{2y} \\ 0 \\ 0 \end{Bmatrix}$$

Hence reduced set of equations to solve for unknown displacements at node 2

$$\frac{EA}{2L}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

Step 5: Solve for unknown displacements

$$\begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix} = \begin{Bmatrix} \dfrac{P_1 L}{EA} \\ \dfrac{P_2 L}{EA} \end{Bmatrix}$$

Step 6: Obtain stresses in the elements

For element #1:

$$\sigma^{(1)} = \frac{E}{L}\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

($d_{1x} \to 0$, $d_{1y} \to 0$)

$$= \frac{E}{\sqrt{2}L}(d_{2x} + d_{2y}) = \frac{P_1 + P_2}{A\sqrt{2}}$$

For element #2:

$$\sigma^{(2)} = \frac{E}{L}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{Bmatrix} d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$

($d_{3x} \to 0$, $d_{3y} \to 0$)

$$= \frac{E}{\sqrt{2}L}(d_{2x} - d_{2y}) = \frac{P_1 - P_2}{A\sqrt{2}}$$

Multi-point constraints

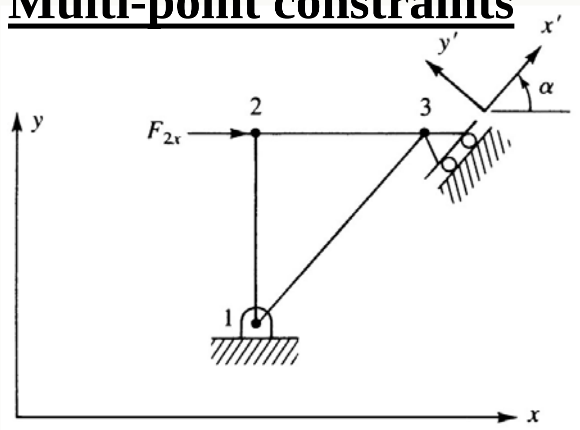
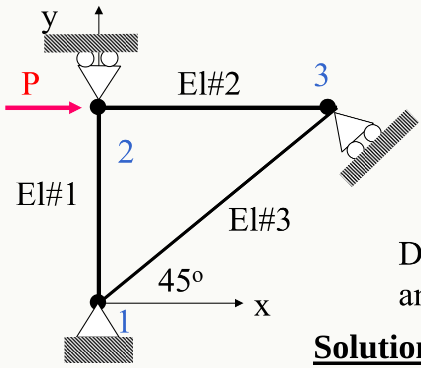


Figure 3-19 Plane truss with inclined boundary conditions at node 3 (see problem worked out in class)

Problem 3: For the plane truss



P=1000 kN,
L=length of elements 1 and 2 = 1m
E=210 GPa
$A = 6\times10^{-4}m^2$ for elements 1 and 2
$= 6\sqrt{2}\times10^{-4}$ m2 for element 3

Determine the unknown displacements and reaction forces.

Solution

Step 1: Node element connectivity table

ELEMENT	Node 1	Node 2
1	1	2
2	2	3
3	1	3

Table of nodal coordinates

Node	x	y
1	0	0
2	0	L
3	L	L

Table of direction cosines

ELEMENT	Length	$l=\dfrac{x_2-x_1}{length}$	$m=\dfrac{y_2-y_1}{length}$
1	L	0	1
2	L	1	0
3	$L\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$

Step 2: Stiffness matrix of each element in global coordinates with global numbering

Stiffness matrix of element 1

$$\underline{k}^{(1)} = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$= \frac{(210\times10^9)(6\times10^{-4})}{1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

(columns: d_{1x}, d_{1y}, d_{2x}, d_{2y}; rows: d_{1x}, d_{1y}, d_{2x}, d_{2y})

Stiffness matrix of element 2

$$\underline{k}^{(2)} = \frac{(210\times10^{9})(6\times10^{-4})}{1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(columns: d_{2x}, d_{2y}, d_{3x}, d_{3y}; rows: d_{2x}, d_{2y}, d_{3x}, d_{3y})

Stiffness matrix of element 3

$$\underline{k}^{(3)} = \frac{(210\times10^{9})(6\sqrt{2}\times10^{-4})}{\sqrt{2}}\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

(columns: d_{1x}, d_{1y}, d_{3x}, d_{3y}; rows: d_{1x}, d_{1y}, d_{3x}, d_{3y})

Step 3: Assemble the global stiffness matrix

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$$\underline{K} = 1260\times10^{5}\begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 1.5 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.5 & -0.5 & -1 & 0 & 1.5 & 0.5 \\ -0.5 & -0.5 & 0 & 0 & 0.5 & 0.5 \end{bmatrix} \text{ N/m}$$

The final set of equations is $\underline{K}\,\underline{d} = \underline{F}$ Eq(1)

Step 4: Incorporate boundary conditions

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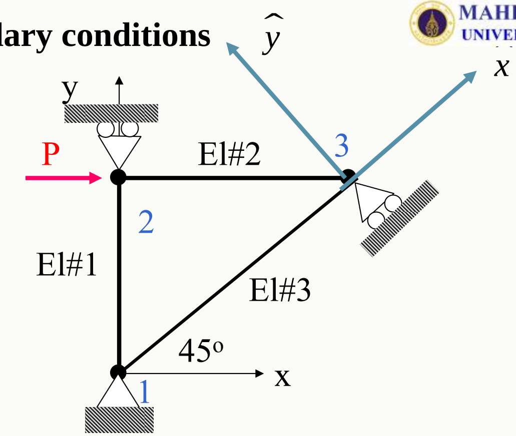
$$\underline{d} = \begin{Bmatrix} 0 \\ 0 \\ d_{2x} \\ 0 \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$



Also, $\hat{d}_{3y} = 0$ in the local coordinate system of element 3

How do I convert this to a boundary condition in the global (x,y) coordinates?

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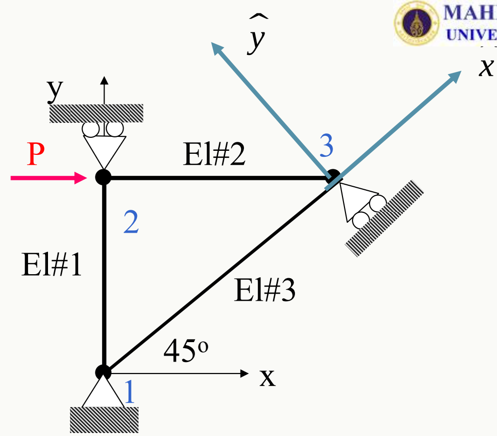
$$\underline{F} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ P \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$



Also, $\hat{F}_{3x} = 0$ in the local coordinate system of element 3

How do I convert this to a boundary condition in the global (x,y) coordinates?

Using coordinate transformations

$$\begin{Bmatrix} \hat{d}_{3x} \\ \hat{d}_{3y} \end{Bmatrix} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \end{Bmatrix} \quad l = m = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \begin{Bmatrix} \hat{d}_{3x} \\ \hat{d}_{3y} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{\sqrt{2}}(d_{3x} + d_{3y}) \\ \frac{1}{\sqrt{2}}(d_{3y} - d_{3x}) \end{Bmatrix}$$

$\hat{d}_{3y} = 0$ (Multi-point constraint)

$$\Rightarrow \hat{d}_{3y} = \frac{1}{\sqrt{2}}(d_{3y} - d_{3x}) = 0$$

$$\Rightarrow d_{3y} - d_{3x} = 0 \qquad \text{Eq (2)}$$

Similarly for the forces at node 3

$$\begin{Bmatrix} \hat{F}_{3x} \\ \hat{F}_{3y} \end{Bmatrix} = \begin{bmatrix} l & m \\ -m & n \end{bmatrix} \begin{Bmatrix} F_{3x} \\ F_{3y} \end{Bmatrix} \quad l = m = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \begin{Bmatrix} \hat{F}_{3x} \\ \hat{F}_{3y} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{\sqrt{2}}(F_{3x} + F_{3y}) \\ \frac{1}{\sqrt{2}}(F_{3y} - F_{3x}) \end{Bmatrix}$$

$\hat{F}_{3x} = 0$

$$\Rightarrow \hat{F}_{3x} = \frac{1}{\sqrt{2}}(F_{3y} + F_{3x}) = 0$$

$$\Rightarrow F_{3y} + F_{3x} = 0 \qquad \text{Eq (3)}$$

Therefore we need to solve the following equations simultaneously

$$\underline{K}\,\underline{d} = \underline{F} \qquad \text{Eq(1)}$$

$$d_{3y} - d_{3x} = 0 \qquad \text{Eq(2)}$$

$$F_{3y} + F_{3x} = 0 \qquad \text{Eq(3)}$$

Incorporate boundary conditions and reduce Eq(1) to

$$1260 \times 10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \\ d_{3y} \end{Bmatrix} = \begin{Bmatrix} P \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

Write these equations out explicitly

$$1260 \times 10^5 (d_{2x} - d_{3x}) = P \qquad \text{Eq(4)}$$

$$1260 \times 10^5 (-d_{2x} + 1.5 d_{3x} + 0.5 d_{3y}) = F_{3x} \qquad \text{Eq(5)}$$

$$1260 \times 10^5 (0.5 d_{3x} + 0.5 d_{3y}) = F_{3y} \qquad \text{Eq(6)}$$

Add Eq (5) and (6)

$$1260 \times 10^5 (-d_{2x} + 2 d_{3x} + d_{3y}) = F_{3x} + F_{3y} = 0 \text{ using Eq(3)}$$

$$\Rightarrow 1260 \times 10^5 (-d_{2x} + 3 d_{3x}) = 0 \quad \text{using Eq(2)}$$

$$\Rightarrow d_{2x} = 3 d_{3x} \quad \text{Eq(7)}$$

Plug this into Eq(4)

$$\Rightarrow 1260 \times 10^5 (3 d_{3x} - d_{3x}) = P$$

$$\Rightarrow 2520 \times 10^5 d_{3x} = 10^6$$

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$$\Rightarrow d_{3x} = 0.003968\,m$$
$$d_{2x} = 3d_{3x} = 0.0119\,m$$

Compute the reaction forces

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = 1260 \times 10^5 \begin{bmatrix} 0 & -0.5 & -0.5 \\ 0 & -0.5 & -0.5 \\ 0 & 0 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$

$$= \begin{Bmatrix} -500 \\ -500 \\ 0 \\ -500 \\ 500 \end{Bmatrix} kN$$

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Physical significance of the stiffness matrix

In general, we will have a stiffness matrix of the form

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

And the finite element force-displacement relation

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

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Physical significance of the stiffness matrix

The first equation is

$$k_{11}d_1 + k_{12}d_2 + k_{13}d_3 = F_1$$

Force equilibrium equation at node 1

Columns of the global stiffness matrix

What if $d_1=1$, $d_2=0$, $d_3=0$?

While **d.o.f** 2 and 3 are held fixed

$F_1 = k_{11}$ Force along **d.o.f** 1 due to unit displacement at **d.o.f** 1

$F_2 = k_{21}$ Force along **d.o.f** 2 due to unit displacement at **d.o.f** 1

$F_3 = k_{31}$ Force along **d.o.f** 3 due to unit displacement at **d.o.f** 1

Similarly we obtain the physical significance of the other entries of the global stiffness matrix

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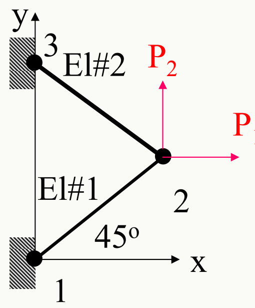
In general

k_{ij} = Force at d.o.f 'i' due to **unit displacement** at d.o.f 'j' keeping **all the other d.o.fs fixed**

Example



The length of bars 12 and 23 are equal (L)
E: Young's modulus
A: Cross sectional area of each bar
Solve for d_{2x} and d_{2y} using the "physical interpretation" approach

Solution

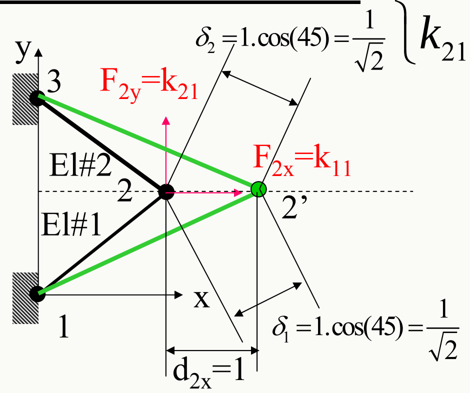
Notice that the final set of equations will be of the form

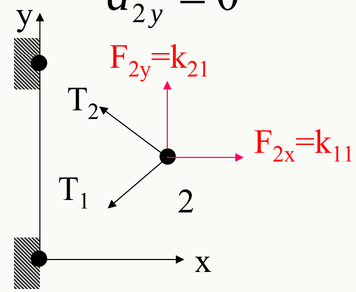
$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

Where k_{11}, k_{12}, k_{21} and k_{22} will be determined using the "physical interpretation" approach

---

To obtain the first column $\begin{Bmatrix} k_{11} \\ k_{21} \end{Bmatrix}$ apply $d_{2x} = 1$, $d_{2y} = 0$





Force equilibrium

$$\sum F_x = k_{11} - T_1\cos(45) - T_2\cos(45) = 0$$
$$\sum F_y = k_{21} - T_1\sin(45) + T_2\sin(45) = 0$$

Force-deformation relations

$$T_1 = \frac{EA}{L}\delta_1$$
$$T_2 = \frac{EA}{L}\delta_2$$

Combining force equilibrium and force-deformation relations

$$k_{11} = \frac{(T_1 + T_2)}{\sqrt{2}} = \frac{EA}{\sqrt{2}L}(\delta_1 + \delta_2)$$
$$k_{21} = \frac{(T_1 - T_2)}{\sqrt{2}} = \frac{EA}{\sqrt{2}L}(\delta_1 - \delta_2)$$

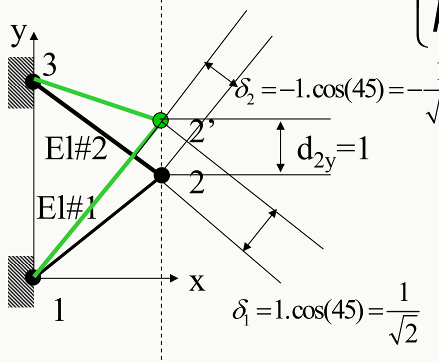
Now use the geometric (compatibility) conditions (see figure)

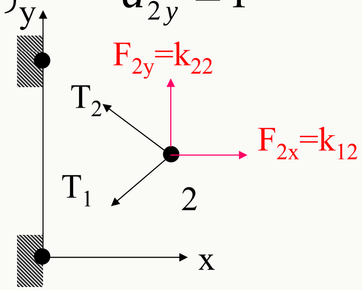
$$\delta_1 = 1.\cos(45) = \frac{1}{\sqrt{2}}$$
$$\delta_2 = 1.\cos(45) = \frac{1}{\sqrt{2}}$$

Finally

$$k_{11} = \frac{EA}{\sqrt{2}L}(\delta_1 + \delta_2) = \frac{EA}{\sqrt{2}L}\left(\frac{2}{\sqrt{2}}\right) = \frac{EA}{L}$$
$$k_{21} = \frac{EA}{\sqrt{2}L}(\delta_1 - \delta_2) = 0$$

To obtain the second column $\begin{Bmatrix} k_{12} \\ k_{22} \end{Bmatrix}$ apply $d_{2x} = 0$, $d_{2y} = 1$





Force equilibrium

$$\sum F_x = k_{12} - T_1\cos(45) - T_2\cos(45) = 0$$
$$\sum F_y = k_{22} - T_1\sin(45) + T_2\sin(45) = 0$$

Force-deformation relations

$$T_1 = \frac{EA}{L}\delta_1$$
$$T_2 = \frac{EA}{L}\delta_2$$

Combining force equilibrium and force-deformation relations

$$k_{12} = \frac{(T_1 + T_2)}{\sqrt{2}} = \frac{EA}{\sqrt{2}L}(\delta_1 + \delta_2)$$

$$k_{22} = \frac{(T_1 - T_2)}{\sqrt{2}} = \frac{EA}{\sqrt{2}L}(\delta_1 - \delta_2)$$

Now use the geometric (compatibility) conditions (see figure)

$$\delta_1 = 1.\cos(45) = \frac{1}{\sqrt{2}}$$

$$\delta_2 = -1.\cos(45) = -\frac{1}{\sqrt{2}}$$

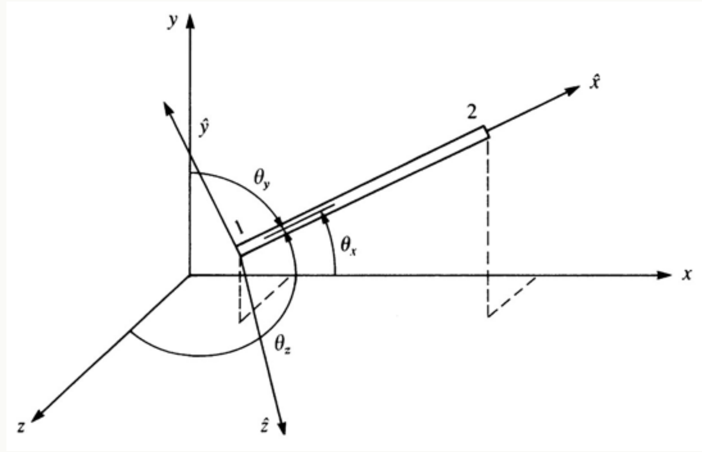
This negative is due to **compression**

Finally

$$k_{12} = \frac{EA}{\sqrt{2}L}(\delta_1 + \delta_2) = 0$$

$$k_{22} = \frac{EA}{\sqrt{2}L}(\delta_1 - \delta_2) = \frac{EA}{\sqrt{2}L}\left(\frac{2}{\sqrt{2}}\right) = \frac{EA}{L}$$

3D Truss (space truss)



In local coordinate system $\underline{\hat{f}} = \underline{\hat{k}}\ \underline{\hat{d}}$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{1z} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{f}_{2z} \end{Bmatrix} = \begin{bmatrix} k & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{1z} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \\ \hat{d}_{2z} \end{Bmatrix}$$

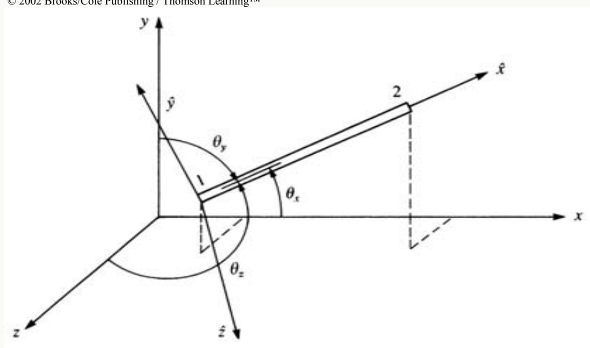
The transformation matrix for a **single vector** in 3D

$$\underline{\hat{d}} = \underline{T}^* \underline{d}$$

$$\underline{T}^* = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

l_1, m_1 *and* n_1 are the direction cosines of $\hat{x}$

$$l_1 = \cos\theta_x$$
$$m_1 = \cos\theta_y$$
$$n_1 = \cos\theta_z$$



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Transformation matrix **T** relating the local and global displacement and load vectors of the truss element

$$\hat{\underline{d}} = \underline{T}\underline{d}$$

$$\hat{\underline{f}} = \underline{T}\underline{f}$$

$$\underset{6\times6}{\underline{T}} = \begin{bmatrix} \underline{T}^* & \underline{0} \\ \underline{0} & \underline{T}^* \end{bmatrix}$$

Element stiffness matrix in global coordinates

$$\underset{6\times6}{\underline{k}} = \underset{6\times6}{\underline{T}^T}\ \underset{6\times6}{\hat{\underline{k}}}\ \underset{6\times6}{\underline{T}}$$

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$$\underline{k} = \underline{T}^T \hat{\underline{k}} \underline{T} = \frac{EA}{L}\begin{bmatrix} l_1^2 & l_1m_1 & l_1n_1 & -l_1^2 & -l_1m_1 & -l_1n_1 \\ l_1m_1 & m_1^2 & m_1n_1 & -l_1m_1 & -m_1^2 & -m_1n_1 \\ l_1n_1 & m_1n_1 & n_1^2 & l_1n_1 & m_1n_1 & -n_1^2 \\ -l_1^2 & -l_1m_1 & -l_1n_1 & l_1^2 & l_1m_1 & l_1n_1 \\ -l_1m_1 & -m_1^2 & -m_1n_1 & l_1m_1 & m_1^2 & m_1n_1 \\ -l_1n_1 & -m_1n_1 & -n_1^2 & l_1n_1 & m_1n_1 & n_1^2 \end{bmatrix}$$

Notice that the direction cosines of **only** the local $\hat{x}$ axis enter the $\underline{k}$ matrix

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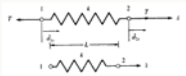
STEP	Linear spring	Bar/Truss
Select element type	 Linear spring	 Bar/Truss
Select displacement function approx. shape function	$\hat{u} = a_1 + a_2\hat{x}$; $\hat{u} = \left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\right)\hat{x} + \hat{d}_{1x}$ ← ; $\hat{u} = [N_1 \quad N_2]\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{2x}\end{Bmatrix}$; $N_1 = 1 - \frac{\hat{x}}{L}$; $N_2 = \frac{\hat{x}}{L}$	Solve for a_1 and a_2: $\hat{u}(\hat{x}=0) = \hat{d}_{1x} = a_1$; $\hat{u}(\hat{x}=L) = \hat{d}_{2x} = a_2 L + a_1$
Define relationships	$T = k\delta$; $\delta = \hat{d}_{2x} - \hat{d}_{1x}$	$\varepsilon_x = \frac{d\hat{u}}{d\hat{x}} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$; $T = A\sigma_x$, $\sigma_x = E\varepsilon_x$
Derive stiffness matrix (Local)	$\begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{2x}\end{Bmatrix} = \begin{bmatrix}k & -k\\ -k & k\end{bmatrix}\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{2x}\end{Bmatrix}$	$\begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{2x}\end{Bmatrix} = \frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{2x}\end{Bmatrix}$
Assemble global stiffness matrix + BC	$\mathbf{K} = [K] = \sum_{e=1}^{N} \hat{k}^{(e)}$;	$\mathbf{F} = \{F\} = \sum_{e=1}^{N} \hat{f}^{(e)}$
Solve for nodal disp. + element forces	$\mathbf{F} = \mathbf{Kd}$; $T = k\delta$	$\mathbf{F} = \mathbf{Kd}$; $T = A\sigma_x$

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• Direct Stiffness – beam

Summary:

- **The principles of simple beam theory**
- **Stiffness matrix of a beam element**
- **Procedures for handling distributed loading and concentrated nodal loading**
- **Example Problems**

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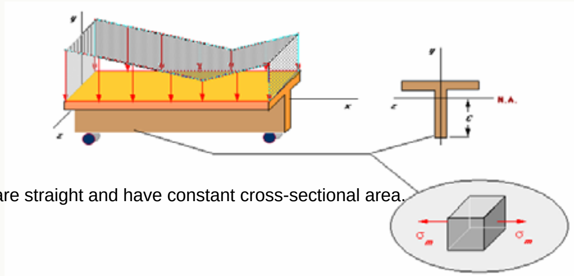
Beams: Engineering structures that are long, slender and generally subjected to transverse loading that produces significant bending effects as opposed to twisting or axial effects



Ideal beams are straight and have constant cross-sectional area.

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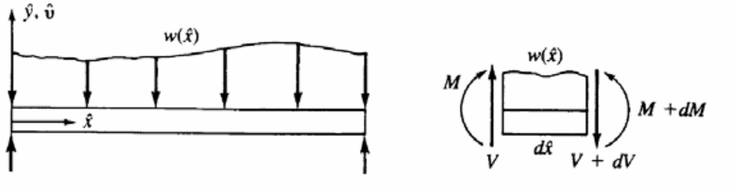
Development of Beam Equations

The differential equation governing simple linear-elastic beam behavior can be derived as follows. Consider the beam shown below.



(a) Beam under load $w(\hat{x})$ (b) Differential beam element

Write the equations of equilibrium for the differential element:

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$$\sum M_{right-side} = 0 = (M + dM) - M - Vd\hat{x} + w(\hat{x})d\hat{x}\left(\frac{d\hat{x}}{2}\right) \qquad d\hat{x}^2 \approx 0$$

$$\sum F_y = 0 = V - (V + dV) - w(\hat{x})dx$$

From force and moment equilibrium of a differential beam element, we get:

$$\sum M_{right-side} = 0 \quad \Rightarrow \quad -Vd\hat{x} + dM = 0 \quad \text{or} \quad V = \frac{dM}{d\hat{x}}$$

$$\sum F_y = 0 \quad \Rightarrow \quad -wd\hat{x} - dV = 0 \quad \text{or} \quad w = -\frac{dV}{d\hat{x}} \quad \Rightarrow \quad w = -\frac{d}{d\hat{x}}\left(\frac{dM}{d\hat{x}}\right)$$

The curvature ***k*** of the beam is related to the moment by:

where ρ is the radius of the deflected curve, $\hat{v}$ is the transverse displacement function in the $\hat{y}$ direction, ***E*** is the modulus of elasticity, and ***I*** is the principle moment of inertia about $\hat{y}$ direction

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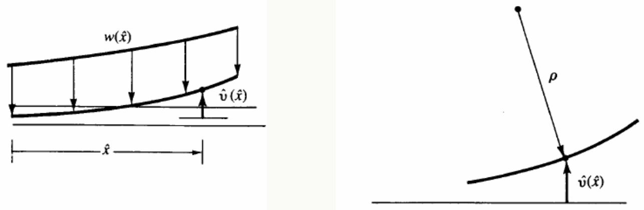

(a) Portion of deflected curve of beam b) Radius of deflected curve at $\hat{v}(\hat{x})$

The curvature for small slopes $\theta = d\hat{v}/d\hat{x}$ is given as:

$$k = \frac{d^2\hat{v}}{d\hat{x}^2}$$

Therefore:

$$\frac{d^2\hat{v}}{d\hat{x}^2} = \frac{M}{EI} \quad \Rightarrow \quad M = EI\frac{d^2\hat{v}}{d\hat{x}^2}$$

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Substituting the moment expression into the moment-load equations gives:

$$\frac{d^2}{d\hat{x}^2}\left(EI\frac{d^2\hat{v}}{d\hat{x}^2}\right) = -w(\hat{x})$$

For constant values of ***EI***, the above equation reduces to:

$$EI\left(\frac{d^4\hat{v}}{d\hat{x}^4}\right) = -w(\hat{x})$$

Stiffness Matrix
A Beam Element

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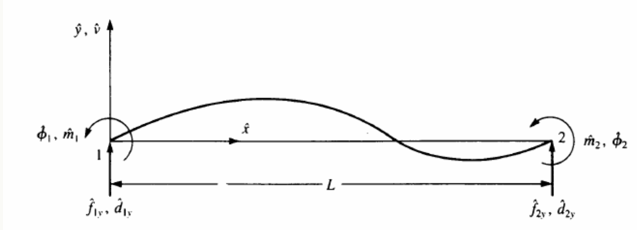
STEP 1: Select Element Type

Consider a linear-elastic beam element shown below



- $\hat{x}$ axial local coordinate
- $\hat{y}$ transverse local coordinate
- $\hat{d}_{iy}$ local transverse nodal
- $\hat{\phi}_i$ the rotations
- L beam length
- $\hat{m}_i$ the bending moments
- $\hat{f}_{iy}$ The local nodal forces

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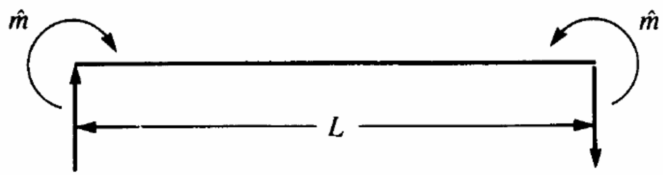
At all nodes, the following sign conventions are used:

1. ***Moments*** are positive in the counterclockwise direction.
2. ***Rotations*** are positive in the counterclockwise direction.
3. ***Forces*** are positive in the positive $\hat{y}$ direction.
4. ***Displacements*** are positive in the positive $\hat{y}$ direction.



Neglect all axial effects!

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STEP 2: Select a displacement function

Assume the transverse displacement function v is

$$v = a_1\hat{x}^3 + a_2\hat{x}^2 + a_3\hat{x} + a_4$$

The number of coefficients in the displacement function, ***a**$_i$*, is equal to the total number of degrees of freedom associated with the element (displacement and rotation at each node). The boundary conditions are:

$$\hat{v}(\hat{x}=0) = \hat{d}_{1y} \qquad \hat{v}(\hat{x}=L) = \hat{d}_{2y}$$

$$\frac{d\hat{v}(\hat{x}=0)}{d\hat{x}} = \hat{\phi}_1 \qquad \frac{d\hat{v}(\hat{x}=L)}{d\hat{x}} = \hat{\phi}_2$$

Applying the boundary conditions

$$\hat{v}(0) = \hat{d}_{1y} = a_4$$

$$\hat{v}(L) = \hat{d}_{2y} = a_1 L^3 + a_2 L^2 + a_3 L + a_4$$

$$\frac{d\hat{v}(0)}{d\hat{x}} = \hat{\phi}_1 = a_3$$

$$\frac{d\hat{v}(L)}{dx} = \hat{\phi}_2 = 3a_1 L^2 + 2a_2 L + a_3$$

Solving these equations for the unknown coefficient gives

$$\hat{v} = \left[\frac{2}{L^3}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) + \frac{1}{L^2}\left(\hat{\phi}_1 - \hat{\phi}_2\right)\right]\hat{x}^3 + \left[-\frac{3}{L^2}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) - \frac{1}{L}\left(2\hat{\phi}_1 + \hat{\phi}_2\right)\right]\hat{x}^2 + \hat{\phi}_1\hat{x} + \hat{d}_{1y}$$

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$$\hat{v} = \left[\frac{2}{L^3}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) + \frac{1}{L^2}\left(\hat{\phi}_1 - \hat{\phi}_2\right)\right]\hat{x}^3 + \left[-\frac{3}{L^2}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) - \frac{1}{L}\left(2\hat{\phi}_1 + \hat{\phi}_2\right)\right]\hat{x}^2 + \hat{\phi}_1\hat{x} + \hat{d}_{1y}$$

In matrix form the above equations are: $\hat{v} = [N]\{\hat{d}\}$

where $\{\hat{d}\} = \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix}$ $\quad [N] = [N_1 \; N_2 \; N_3 \; N_4]$

Shape Functions for a Beam Element

and

$$N_1 = \frac{1}{L^3}\left(2\hat{x}^3 - 3\hat{x}^2 L + L^3\right) \qquad N_2 = \frac{1}{L^3}\left(\hat{x}^3 L - 2\hat{x}^2 L^2 + \hat{x} L^3\right)$$

$$N_3 = \frac{1}{L^3}\left(-2\hat{x}^3 + 3\hat{x}^2 L\right) \qquad N_4 = \frac{1}{L^3}\left(\hat{x}^3 L - \hat{x}^2 L^2\right)$$

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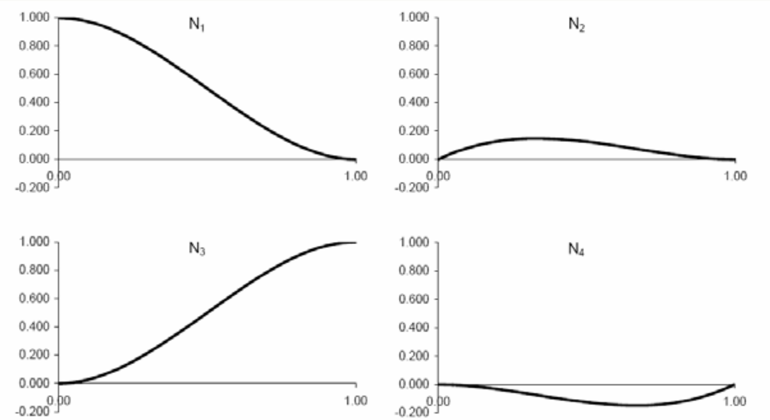
Shape Functions for a Beam Element



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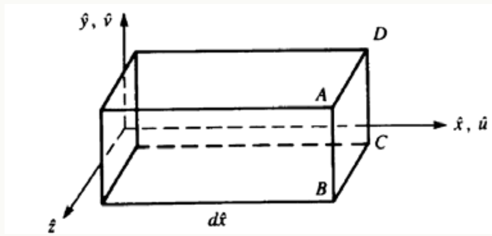
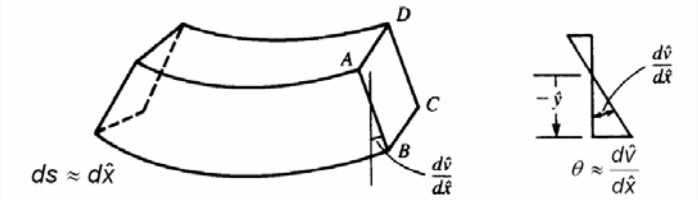
STEP 3: Define the strain/displacement + stress/strain relationships

The stress-displacement relationship is: $\varepsilon_x(\hat{x}, \hat{y}) = \frac{d\hat{u}}{d\hat{x}}$

where $\hat{u}$ is the axial displacement function.

We can relate the axial displacement to the transverse displacement by considering the beam element shown below:





$\hat{u} = -\hat{y}\frac{d\hat{v}}{d\hat{x}}$ One of the basic assumptions in simple beam theory is that planes remain planar after deformation, therefore:

$$\varepsilon_x(\hat{x},\hat{y}) = -\hat{y}\left(\frac{d^2\hat{v}}{d\hat{x}^2}\right)$$

Moments and shears are related to the transverse displacement as:

$$\hat{m}(\hat{x}) = EI\left(\frac{d^2\hat{v}}{d\hat{x}^2}\right) \qquad \hat{V}(x) = EI\left(\frac{d^3\hat{v}}{d\hat{x}^3}\right)$$

STEP 4: Derive the element stiffness matrix and equations

Using beam theory sign convention for shear force and bending moment, one obtain the following equations:

$$\hat{f}_{1y} = \hat{V} = EI\frac{d^3\hat{v}(0)}{d\hat{x}^3} = \frac{EI}{L^3}\left(12\hat{d}_{1y} + 6L\hat{\phi}_1 - 12\hat{d}_{2y} + 6L\hat{\phi}_2\right)$$

$$\hat{f}_{2y} = -\hat{V} = EI\frac{d^3\hat{v}(L)}{d\hat{x}^3} = \frac{EI}{L^3}\left(-12\hat{d}_{1y} - 6L\hat{\phi}_1 + 12\hat{d}_{2y} - 6L\hat{\phi}_2\right)$$

$$\hat{m}_1 = -\hat{m} = -EI\frac{d^2\hat{v}(0)}{d\hat{x}^2} = \frac{EI}{L^3}\left(6L\hat{d}_{1y} + 4L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 2L^2\hat{\phi}_2\right)$$

$$\hat{m}_2 = \hat{m} = EI\frac{d^2\hat{v}(L)}{d\hat{x}^2} = \frac{EI}{L^3}\left(6L\hat{d}_{1y} + 2L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 4L^2\hat{\phi}_2\right)$$

In the matrix form the above equations are:

$$\begin{Bmatrix}\hat{f}_{1y}\\ \hat{m}_1\\ \hat{f}_{2y}\\ \hat{m}_2\end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}\begin{Bmatrix}\hat{d}_{1y}\\ \hat{\phi}_1\\ \hat{d}_{2y}\\ \hat{\phi}_2\end{Bmatrix}$$

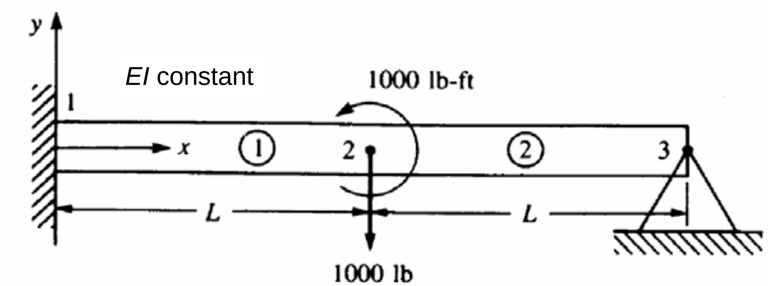
Where the stiffness matrix is:

$$k = \frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}$$

STEP 5: Assemble the element equations and Introduce boundary conditions

This will be illustrated in the following example!

Consider a beam modeled by two beam elements, shown below:



The beam element stiffness matrices are:

$$k^{(1)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (d_{1y},\ \phi_1,\ d_{2y},\ \phi_2)$$

$$k^{(2)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (d_{2y},\ \phi_2,\ d_{3y},\ \phi_3)$$

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In this example, the local coordinates coincide with the global coordinates of the whole beam (therefore there is no transformation required for this problem). The total stiffness matrix can be assembled as:

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

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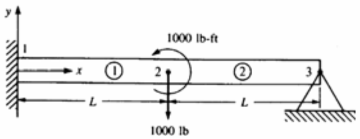
STEP 6: Introduce boundary conditions

The boundary conditions are:

$$d_{1y} = \phi_1 = d_{3y} = 0$$



By applying the boundary conditions the beam equations reduce to:

$$\begin{Bmatrix} -1{,}000\ lb \\ 1{,}000\ lb \cdot ft \\ 0 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

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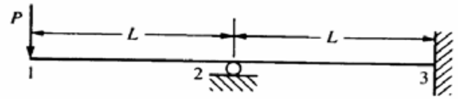
Solving the above equations gives:

$$\begin{Bmatrix} d_{2y} \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{875L^3 - 375L^2}{12EI}\ lb \\ -\dfrac{125L^2 - 625L}{4EI}\ rad \\ -\dfrac{125L^2 - 125L}{EI}\ rad \end{Bmatrix}$$

Example 1 - Beam Problem

Consider the beam shown below. Assume that ***EI*** is constant and the length is 2***L***.



The beam element stiffness matrices are:

$$k^{(1)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (\text{columns: } d_{1y}, \phi_1, d_{2y}, \phi_2)$$

$$k^{(2)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (\text{columns: } d_{2y}, \phi_2, d_{3y}, \phi_3)$$

The local coordinates coincide with the global coordinates of the whole beam (therefore there is no transformation required for this problem). The total stiffness matrix can be assembled as:

$$K = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

The governing beam equations are:

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

The boundary conditions are:

$$d_{2y} = d_{3y} = \phi_3 = 0$$

By applying the boundary conditions the beam equations reduce to:

$$\begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ \phi_2 \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} d_{1y} \\ \phi_1 \\ \phi_2 \end{Bmatrix} = \frac{PL^2}{4EI}\begin{Bmatrix} -\frac{7L}{3} \\ 3 \\ 1 \end{Bmatrix}$$

The positive signs for the rotations indicate that both are in the counterclockwise direction. The negative sign on the displacement indicates a deformation in the $-\hat{y}$ direction.

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{P}{4L}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -7L/3 \\ 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

The local nodal forces for element 1:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{P}{4L}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -7L/3 \\ 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \\ P \\ -PL \end{Bmatrix}$$

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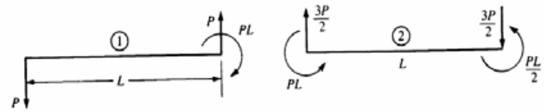
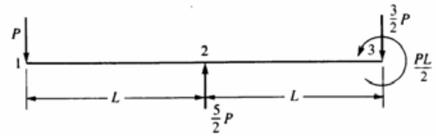
The local nodal forces for element 2:

$$\begin{Bmatrix} \hat{f}_{2y} \\ \hat{m}_2 \\ \hat{f}_{3y} \\ \hat{m}_3 \end{Bmatrix} = \frac{P}{4L}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.5P \\ PL \\ -1.5P \\ 0.5PL \end{Bmatrix}$$

The free-body diagrams for the each element are shown below.



Combining the elements gives the forces and moments for the original beam.



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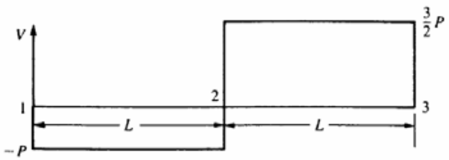
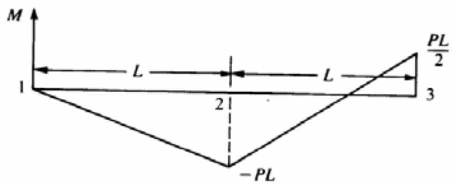
Therefore, the shear force and bending moment diagrams are:





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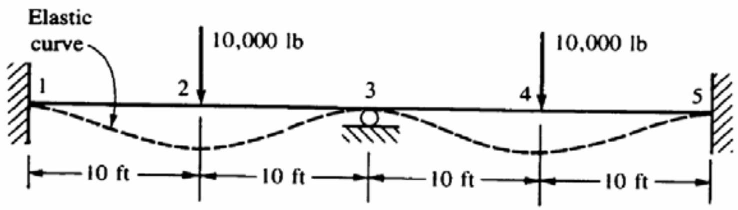
Example 2 - Beam Problem

Consider the beam shown below. Assume ***E*** = 30 x 10^6 psi and ***I*** = 500 in.^{4} are constant throughout the beam. Use four elements of equal length to model the beam.



The beam element stiffness matrices are:

$$k^{(i)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad \text{(columns: } d_{iy},\ \phi_i,\ d_{(i+1)y},\ \phi_{i+1}\text{)}$$

Using the direct stiffness method, the four beam element stiffness matrices are superimposed to produce the global stiffness matrix. As shown on the next page.

The boundary conditions for this problem are:

$$d_{1y} = \phi_1 = d_{3y} = d_{5y} = \phi_5 = 0$$

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Element 1

Element 2

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \\ M_4 \\ F_{5y} \\ M_5 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L & 0 & 0 & 0 & 0 \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -6L & 12+12 & -6L+6L & -12 & 6L & 0 & 0 \\ 0 & 0 & 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 0 & 0 & 0 & 0 & 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 0 & 0 & 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \\ d_{4y} \\ \phi_4 \\ d_{5y} \\ \phi_5 \end{Bmatrix}$$

(columns: d_{1y}, ϕ_1, d_{2y}, ϕ_2, d_{3y}, ϕ_3, d_{4y}, ϕ_4, d_{5y}, ϕ_5)

Element 3

Element 4

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After applying the boundary conditions the global beam equations reduce to:

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 0 & 6L & 0 & 0 \\ 0 & 8L^2 & 2L^2 & 0 & 0 \\ 6L & 2L^2 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -6L & 24 & 0 \\ 0 & 0 & 2L^2 & 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ \phi_3 \\ d_{4y} \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} -10{,}000\ lb \\ 0 \\ 0 \\ -10{,}000\ lb \\ 0 \end{Bmatrix}$$

Substituting **L** = 120 in., **E** = 30 x 10^6 psi, and **I** = 500 in.^{4} into the above equations and solving for the unknowns gives:

$$d_{2y} = d_{4y} = -0.048\ in \qquad \phi_2 = \phi_3 = \phi_4 = 0$$

The global forces and moments can be determined as:

$$F_{1y} = 5\ kips \qquad M_1 = 25\ kips{\cdot}ft$$

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The global forces and moments can be determined as:

$$F_{1y} = 5\ kips \qquad M_1 = 25\ kips{\cdot}ft$$
$$F_{2y} = 10\ kips \qquad M_2 = 0$$
$$F_{3y} = 10\ kips \qquad M_3 = 0$$
$$F_{4y} = 10\ kips \qquad M_4 = 0$$
$$F_{5y} = 5\ kips \qquad M_5 = -25\ kips{\cdot}ft$$

The local nodal forces for element 1:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.048 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 5\ kips \\ 25\ k{\cdot}ft \\ -5\ kips \\ 25\ k{\cdot}ft \end{Bmatrix}$$

The local nodal forces for element 2:

$$\begin{Bmatrix} \hat{f}_{2y} \\ \hat{m}_2 \\ \hat{f}_{3y} \\ \hat{m}_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.048 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -5\ kips \\ -25\ k\text{-}ft \\ 5\ kips \\ -25\ k\text{-}ft \end{Bmatrix}$$

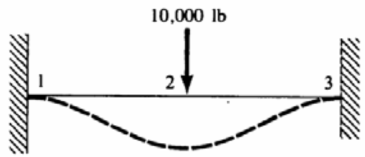
The local nodal forces for element 3:

$$\begin{Bmatrix} \hat{f}_{3y} \\ \hat{m}_3 \\ \hat{f}_{4y} \\ \hat{m}_4 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.048 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 5\ kips \\ 25\ k\text{-}ft \\ -5\ kips \\ 25\ k\text{-}ft \end{Bmatrix}$$

The local nodal forces for element 4:

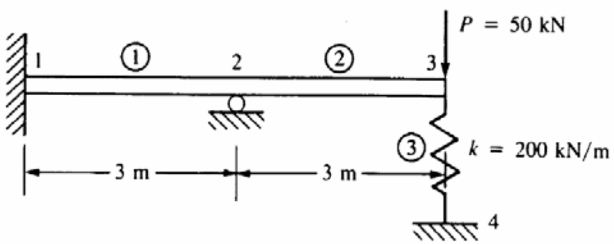
$$\begin{Bmatrix} \hat{f}_{4y} \\ \hat{m}_4 \\ \hat{f}_{5y} \\ \hat{m}_5 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.048 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -5\ kips \\ -25\ k\text{-}ft \\ 5\ kips \\ -25\ k\text{-}ft \end{Bmatrix}$$

Note: Due to symmetry about the vertical plane at node 3, we could have worked just half the beam, as shown below.



Example 3 - Beam Problem

Consider the beam shown below. Assume ***E*** = 210 GPa and ***I*** = 2 x 10^{-4} m^4 are constant throughout the beam and the spring constant ***k*** = 200 kN/m. Use two beam elements of equal length and one spring element to model the structure.



The beam element stiffness matrices are:

$$k^{(1)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (d_{1y},\ \phi_1,\ d_{2y},\ \phi_2)$$

$$k^{(2)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (d_{2y},\ \phi_2,\ d_{3y},\ \phi_3)$$

The spring element stiffness matrix is:

$$k^{(3)} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad (d_{3y},\ d_{4y}) \quad \Rightarrow \quad k^{(3)} = \begin{bmatrix} k & 0 & -k \\ 0 & 0 & 0 \\ -k & 0 & k \end{bmatrix} \quad (d_{3y},\ \phi_3,\ d_{4y})$$

Using the direct stiffness method and superposition gives the global beam equations.

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L & 0 \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 0 \\ 0 & 0 & -12 & -6L & 12+k' & -6L & -k' \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 & 0 \\ 0 & 0 & 0 & 0 & -k' & 0 & k' \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \\ d_{4y} \end{Bmatrix} \qquad k' = \frac{kL^3}{EI}$$

The boundary conditions for this problem are:

$$d_{1y} = \phi_1 = d_{2y} = d_{4y} = 0$$

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After applying the boundary conditions the global beam equations reduce to:

$$\begin{Bmatrix} M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12+k' & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{3PL^2}{EI}\left(\frac{1}{12+7k'}\right) \\ -\frac{7PL^3}{EI}\left(\frac{1}{12+7k'}\right) \\ -\frac{9PL^2}{EI}\left(\frac{1}{12+7k'}\right) \end{Bmatrix} \qquad k' = \frac{kL^3}{EI}$$

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Substituting **L** = 3 m, **E** = 210 GPa, **I** = 2 x 10^{-4} m^4, and **k** = 200 kN/m in the above equations gives:

$$d_{3y} = -0.0174\ m$$
$$\phi_2 = -0.00249\ rad$$
$$\phi_3 = -0.00747\ rad$$

Substituting the solution back into the global equations gives:

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} -69.9\ kN \\ -69.7\ kN \cdot m \\ 116.4\ kN \\ 0 \\ -50\ kN \\ 0 \\ 3.5\ kN \end{Bmatrix}$$

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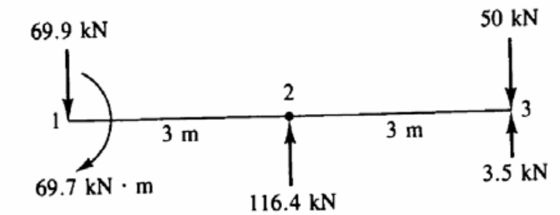
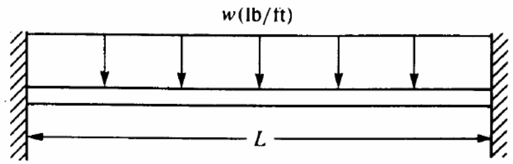
A free-body diagram, including forces and moments acting on the beam is shown below.



Distributed Loadings

Beam members can support distributed loading as well as concentrated nodal loading. Therefore, we must be able to account for distributed loading. Consider the fixed-fixed beam subjected to a uniformly distributed loading ***w*** shown the figure below. The reactions, determined from structural analysis theory, are called fixed-end reactions. In general, fixed-end reactions are those reactions at the ends of an element if the ends of the element are assumed to be fixed (displacements and rotations are zero). Therefore, guided by the results from structural analysis for the case of a uniformly distributed load, we replace the load by concentrated nodal forces and moments tending to have the same effect on the beam as the actual distributed load.



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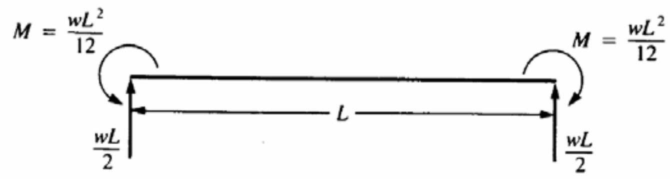

The figure below illustrates the idea of equivalent nodal loads for a general beam. We can replace the effects of a uniform load by a set of nodal forces and moments.

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Work Equivalence Method

This method is based on the concept that the work done by the distributed load is equal to the work done by the discrete nodal loads. The work done by the distributed load is:

$$W_{distributed} = \int_0^L w(\hat{x})\,\hat{v}(\hat{x})\,d\hat{x}$$

where $\hat{v}(\hat{x})$ is the transverse displacement. The work done by the discrete nodal forces is:

$$W_{nodes} = \hat{m}_1\hat{\phi}_1 + \hat{m}_2\hat{\phi}_2 + \hat{f}_{1y}\hat{d}_{1y} + \hat{f}_{2y}\hat{d}_{2y}$$

The nodal forces can be determined by setting $W_{distributed} = W_{nodes}$ for arbitrary displacements and rotations.

$$W_{distributed} = W_{nodes}$$

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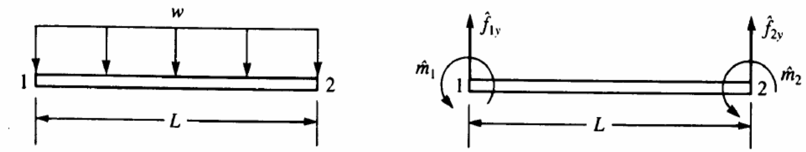
Example 4 - Load Replacement

Consider the beam, shown below, and determine the equivalent nodal forces for the given distributed load.



Using the work equivalence method or:

$$W_{distributed} = W_{nodes}$$

we get:

$$\int_0^L w(\hat{x})\,\hat{v}(\hat{x})\,d\hat{x} = \hat{m}_1\hat{\phi}_1 + \hat{m}_2\hat{\phi}_2 + \hat{f}_{1y}\hat{d}_{1y} + \hat{f}_{2y}\hat{d}_{2y}$$

Evaluating the left-hand-side of the above expression using $w(\hat{x}) = -w$ and $\hat{v}(\hat{x})$ equal to:

$$\hat{v}(\hat{x}) = \left[\frac{2}{L^3}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) + \frac{1}{L^2}\left(\hat{\phi}_1 + \hat{\phi}_2\right)\right]\hat{x}^3 + \left[-\frac{3}{L^2}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) - \frac{1}{L}\left(2\hat{\phi}_1 + \hat{\phi}_2\right)\right]\hat{x}^2 + \hat{\phi}_1\hat{x} + \hat{d}_{1y}$$

gives:

$$\int_0^L w\,\hat{v}(\hat{x})\,d\hat{x} = \frac{Lw}{2}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) - \frac{L^2 w}{4}\left(\hat{\phi}_1 + \hat{\phi}_2\right) - Lw\left(\hat{d}_{2y} - \hat{d}_{1y}\right) + \frac{L^2 w}{3}\left(2\hat{\phi}_1 + \hat{\phi}_2\right) - \frac{L^2 w}{2}\hat{\phi}_1 - wL\hat{d}_{1y}$$

Using a set of arbitrary nodal displacements, such as:

$$d_{1y} = d_{2y} = \phi_2 = 0 \qquad \phi_1 = 1$$

The resulting nodal equivalent force or moment is:

$$\hat{m}_1(1) = -\left(\frac{wL^2}{4} - \frac{2}{3}L^2 w + \frac{L^2}{2}w\right) = -\frac{wL^2}{12}$$

Using another set of arbitrary nodal displacements, such as:

$$d_{1y} = d_{2y} = \phi_1 = 0 \qquad \phi_2 = 1$$

The resulting nodal equivalent force or moment is:

$$\hat{m}_2(1) = -\left(\frac{wL^2}{4} - \frac{wL^2}{3}\right) = \frac{wL^2}{12}$$

Setting the nodal rotations equal zero except for the $\hat{d}_{1y}$ and $\hat{d}_{2y}$ gives:

$$\hat{f}_{1y}(1) = -\frac{LW}{2} + Lw - Lw = -\frac{Lw}{2}$$

$$\hat{f}_{2y}(1) = \frac{LW}{2} - Lw = -\frac{Lw}{2}$$

General Formulation

We can account for the distributed loads or concentrated loads acting on a beam elements by considering the following formulation for a general structure:

$$F = Kd - F_0$$

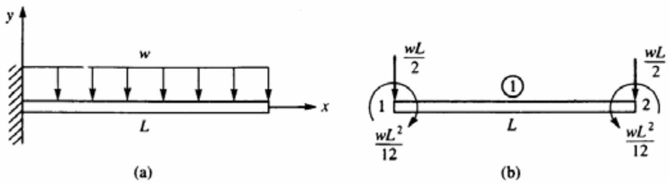
where F_0 are the ***equivalent nodal forces***, expressed in terms of the global-coordinate components. These force would yield the same displacements as the original distributed load. If we assume that the global nodal forces are not present (***F*** = 0) then:

$$F_0 = Kd$$

We now solve for the displacements, ***d***, given the nodal forces $\boldsymbol{F}_0$. Next, substitute the displacements and the equivalent nodal forces $\boldsymbol{F}_0$ back into the original expression and solve for the global nodal forces.

Example 5 - Load Replacement

Consider the beam shown below, determine the equivalent nodal forces for the given distributed load.



The work equivalent nodal forces are shown above. Using the beam stiffness equations, with the boundary conditions applied, we can solve for the displacements

$$\begin{Bmatrix} -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

Therefore:

$$\begin{Bmatrix} \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{wL^4}{8EI} \\ -\dfrac{wL^3}{6EI} \end{Bmatrix}$$

In this case, the method of equivalent nodal forces gives the exact solution for the displacements and rotations.

To obtain the global nodal forces, we will first define the product of ***Kd*** to be ***F***^{e}, where ***F***^{e} is called the ***effective global nodal forces***. Therefore:

$$\begin{Bmatrix} F^e_{1y} \\ M^e_1 \\ F^e_{2y} \\ M^e_1 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -{wL^4}/{8EI} \\ -{wL^3}/{6EI} \end{Bmatrix}$$

Simplifying the above expression gives:

$$\begin{Bmatrix} F^e_{1y} \\ M^e_1 \\ F^e_{2y} \\ M^e_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{wL}{2} \\ \dfrac{5wL^2}{12} \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} \qquad \begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{wL}{2} \\ \dfrac{5wL^2}{12} \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} - \begin{Bmatrix} -\dfrac{wL}{2} \\ -\dfrac{wL^2}{12} \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} = \begin{Bmatrix} wL \\ \dfrac{wL^2}{2} \\ 0 \\ 0 \end{Bmatrix}$$

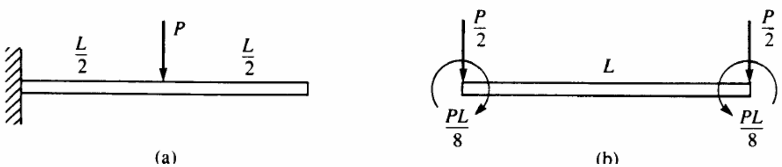
Using the above expression and the fix-end moments in:

$$F = Kd - F_0$$

gives the correct global nodal forces as:

Example 6 - Cantilever Beam

Consider the beam, shown below, determine the vertical displacement and rotation at the free-end and the nodal forces, including reactions. Assume ***EI*** is constant throughout the beam.



We will use one element and replace the concentrated load with the appropriate nodal forces. The beam stiffness equations become:

$$\begin{Bmatrix} -\dfrac{P}{2} \\ \dfrac{PL}{8} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

Therefore:

$$\begin{Bmatrix} \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{5PL^3}{48EI} \\ -\dfrac{PL^2}{8EI} \end{Bmatrix}$$

To obtain the global nodal forces, we begin by evaluating the effective nodal forces.

$$\begin{Bmatrix} F^e_{1y} \\ M^e_1 \\ F^e_{2y} \\ M^e_1 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -{5PL^3}/{48EI} \\ -{PL^2}/{8EI} \end{Bmatrix}$$

Simplifying the above expression gives:

$$\begin{Bmatrix} F^{e}_{1y} \\ M^{e}_{1} \\ F^{e}_{2y} \\ M^{e}_{1} \end{Bmatrix} = \begin{Bmatrix} \frac{P}{2} \\ \frac{3PL}{8} \\ -\frac{P}{2} \\ \frac{PL}{8} \end{Bmatrix}$$

Using the above expression in the following equation, gives:

$$F = Kd - F_0$$

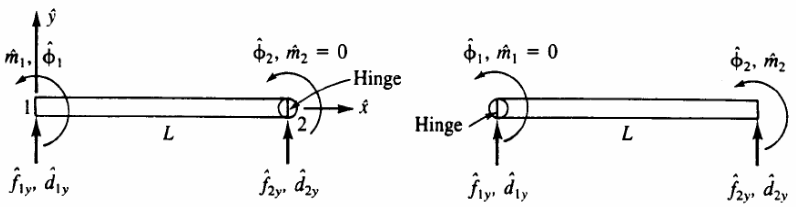
The correct global nodal forces as:

$$\begin{Bmatrix} F_{1y} \\ M_{1} \\ F_{2y} \\ M_{2} \end{Bmatrix} = \begin{Bmatrix} \frac{P}{2} \\ \frac{3PL}{8} \\ -\frac{P}{2} \\ \frac{PL}{8} \end{Bmatrix} - \begin{Bmatrix} -\frac{P}{2} \\ -\frac{PL}{8} \\ -\frac{P}{2} \\ \frac{PL}{8} \end{Bmatrix} = \begin{Bmatrix} P \\ \frac{PL}{2} \\ 0 \\ 0 \end{Bmatrix}$$

In general, for any structure in which an equivalent nodal force replacement is made, the actual nodal forces acting on the structure are determined by first evaluating the effective nodal forces $\boldsymbol{F}^{e}$ for the structure and then subtracting off the equivalent nodal forces $\boldsymbol{F}_0$ for the structure. Similarly, for any element of a structure in which equivalent nodal force replacement is made, the actual local nodal forces acting on the element are determined by first evaluating the effective local nodal forces $\hat{f}^{(e)}$ for the element and then subtracting off the equivalent local nodal forces $\hat{f}_0$ associated only with the element.

Beam Element with Nodal Hinge

Consider the beam, shown below, with an internal hinge. An internal hinge causes a discontinuity in the slope of the deflection curve at the hinge and the bending moment is zero at the hinge.



For a beam with a hinge on the right end, the moment $\hat{m}_2$ is zero and we can partition the matrix to eliminate the degree of freedom associated with $\hat{\phi}_2$.

For a beam with a hinge on the right end, the moment $\hat{m}_2$ is zero and we can partition the matrix to eliminate the degree of freedom associated with $\hat{\phi}_2$.

$$\hat{k} = \frac{EI}{L^3}\left[\begin{array}{ccc:c} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ \hdashline 6L & 2L^2 & -6L & 4L^2 \end{array}\right]$$

We can condense out the degree of freedom by using the partitioning method discussed earlier. Recall, the form of $\boldsymbol{k_c}$

$$k_c = [K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}]$$

$$k_c = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 \\ 6L & 4L^2 & -6L \\ -12 & -6L & 12 \end{bmatrix} - \frac{EI}{L^3}\begin{Bmatrix} 6L \\ 2L^2 \\ -6L \end{Bmatrix}\frac{1}{4L^2}\begin{bmatrix} 6L & 2L^2 & -6L \end{bmatrix}$$

Therefore, the condensed stiffness matrix is:

$$k_c = \frac{3EI}{L^3}\begin{bmatrix} 1 & L & -1 \\ L & L^2 & -L \\ -1 & -L & 1 \end{bmatrix}$$

The element force-displacement equations are:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \end{Bmatrix} = \frac{3EI}{L^3}\begin{bmatrix} 1 & L & -1 \\ L & L^2 & -L \\ -1 & -L & 1 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \end{Bmatrix}$$

Expanding the element force-displacement equations and maintaining $\hat{m}_2 = 0$ gives:

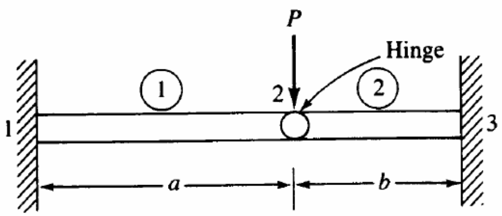
$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{3EI}{L^3}\begin{bmatrix} 1 & L & -1 & 0 \\ L & L^2 & -L & 0 \\ -1 & -L & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

The element force-displacement equations maintaining $\hat{m}_1 = 0$ gives:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{3EI}{L^3}\begin{bmatrix} 1 & 0 & -1 & L \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -L \\ L & 0 & -L & L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

Example 7 - Beam With Hinge

In the following beam, shown below, determine the vertical displacement and rotation at node 2 and the element forces for the uniform beam with an internal hinge at node 2. Assume ***EI*** is constant throughout the beam.



The stiffness matrix for element 1 (with hinge) is:

$$\begin{matrix} d_{1y} & \phi_1 & d_{2y} & \phi_2 \end{matrix}$$

$$k^{(1)} = \frac{3EI}{a^3}\begin{bmatrix} 1 & a & -1 & 0 \\ a & a^2 & -a & 0 \\ -1 & -a & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

The stiffness matrix for element 2 (without hinge) is:

$$\begin{matrix} d_{2y} & \phi_2 & d_{3y} & \phi_3 \end{matrix}$$

$$k^{(2)} = \frac{EI}{b^3}\begin{bmatrix} 12 & 6b & -12 & 6b \\ 6b & 4b^2 & -6b & 2b^2 \\ -12 & -6b & 12 & -6b \\ 6b & 2b^2 & -6b & 4b^2 \end{bmatrix}$$

The boundary conditions for this problem are:

$$d_{1y} = d_{3y} = \phi_1 = \phi_3 = 0$$

After applying the boundary conditions the global beam equations reduce to:

$$EI\begin{bmatrix} \frac{3}{a^3}+\frac{12}{b^3} & \frac{6}{b^2} \\ \frac{6}{b^2} & \frac{4}{b} \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{a^3b^3P}{3\left(b^3+a^3\right)EI} \\ \frac{a^3b^3P}{2\left(b^3+a^3\right)EI} \end{Bmatrix}$$

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The element force-displacement equations for element 1 are:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \end{Bmatrix} = \frac{3EI}{a^3}\begin{bmatrix} 1 & a & -1 \\ a & a^2 & -a \\ -1 & -a & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -\frac{a^3b^3P}{3(b^3+a^3)EI} \end{Bmatrix}$$

Therefore:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \end{Bmatrix} = \begin{Bmatrix} \frac{b^3P}{b^3+a^3} \\ \frac{ab^3P}{b^3+a^3} \\ -\frac{b^3P}{b^3+a^3} \end{Bmatrix}$$

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The element force-displacement equations for element 2 are:

$$\begin{Bmatrix} \hat{f}_{2y} \\ \hat{m}_2 \\ \hat{f}_{3y} \\ \hat{m}_3 \end{Bmatrix} = \frac{EI}{b^3}\begin{bmatrix} 12 & 6b & -12 & 6b \\ 6b & 4b^2 & -6b & 2b^2 \\ -12 & -6b & 12 & -6b \\ 6b & 2b^2 & -6b & 4b^2 \end{bmatrix}\begin{Bmatrix} -\frac{a^3b^3P}{3\left(b^3+a^3\right)EI} \\ \frac{a^3b^2P}{2\left(b^3+a^3\right)EI} \\ 0 \\ 0 \end{Bmatrix}$$

Therefore:

$$\begin{Bmatrix} \hat{f}_{2y} \\ \hat{m}_2 \\ \hat{f}_{3y} \\ \hat{m}_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{a^3P}{b^3+a^3} \\ 0 \\ \frac{a^3P}{b^3+a^3} \\ \frac{ba^3P}{b^3+a^3} \end{Bmatrix}$$

- Many bridges and buildings are composed of frames and grids.

Development of Plane Frame Equations

Rigid Plane Frame

- A rigid plane frame is defined as a series of beam elements rigidly connected to each other.
- The angles made between elements at joints remained unchanged after the deformation.
- Moments are transmitted from one element to another at joints.
- The element centroid and the applied load lie in a common plane.

• Direct Stiffness – plane frame

Summary:

•Local stiffness matrix of a beam element oriented in a plane including axial deformation effects.

The equations & methods for sol. of plane frame.

• Example Problems: frames with inclined and skewed supports

Stiffness matrix of a beam element oriented in a plane

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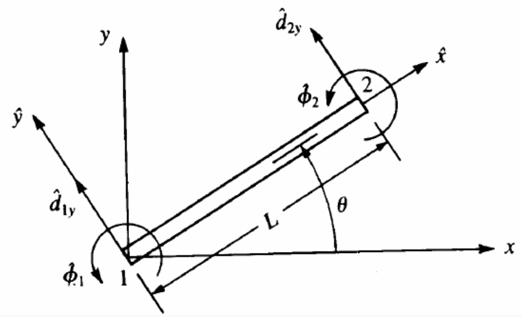
Two-Dimensional Arbitrarily Oriented Beam Element

We can derive the stiffness matrix for an arbitrarily oriented beam element, shown in the figure below, in a manner similar to that used for the bar element. The local axes $\hat{x}$ and $\hat{y}$ are located along the beam element and transverse to the beam element, respectively, and the global axes x and y are located to be convenient for the total structure.



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The transformation from local displacements to global displacements is given in matrix form as:

$$\begin{Bmatrix} \hat{d}_x \\ \hat{d}_y \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} d_x \\ d_y \end{Bmatrix} \qquad \begin{matrix} C = \cos\theta \\ S = \sin\theta \end{matrix}$$

Using the second equation for the beam element, we can relate local nodal degrees of freedom to global degree of freedom:

$$\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} = \begin{bmatrix} -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ \phi_1 \\ d_{2x} \\ d_{2y} \\ \phi_2 \end{Bmatrix} \qquad \hat{d}_y = -Sd_x + Cd_y$$

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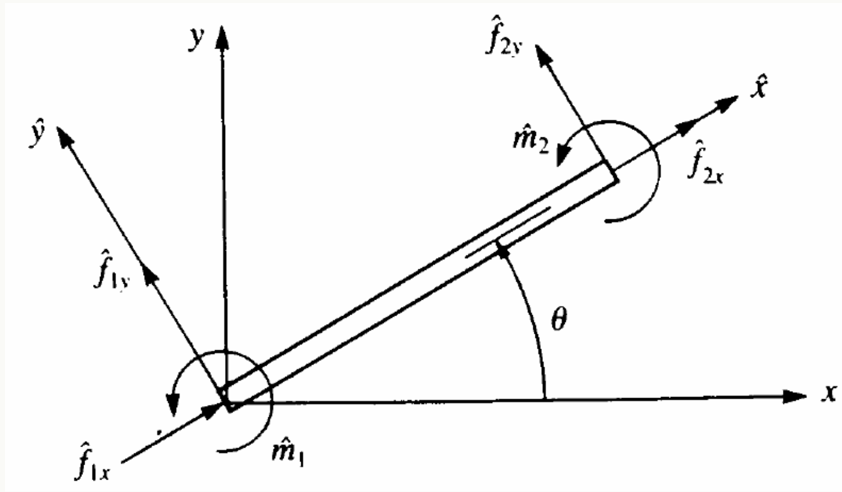
For a beam we will define the following as the ***transformation matrix***:

$$T = \begin{bmatrix} -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that the rotations are not affected by the orientation of the beam. Substituting the above transformation into the general form of the stiffness matrix $k = T^T \hat{k} T$ gives:

$$k = \frac{EI}{L^3} \left[\begin{array}{ccc:ccc} 12S^2 & -12SC & -6LS & -12S^2 & 12SC & -6LS \\ -12SC & 12C^2 & 6LC & 12SC & -12C^2 & 6LC \\ -6LS & 6LC & 4L^2 & 6LS & -6LC & 2L^2 \\ \hdashline -12S^2 & 12SC & 6LS & 12S^2 & -12SC & 6LS \\ 12SC & -12C^2 & -6LC & -12SC & 12C^2 & -6LC \\ -6LS & 6LC & 2L^2 & 6LS & -6LC & 4L^2 \end{array} \right]$$

The effect of axial force in the beam transformation



Recall the simple axial deformation, define in the spring element:

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

Combining the axial effects with the shear force and bending moment effects, in local coordinates,
gives:

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1x} \\ \hat{\phi}_1 \\ \hat{d}_{2x} \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

where

$$C_1 = \frac{AE}{L} \qquad C_2 = \frac{EI}{L^3}$$

Therefore:

$$\hat{k} = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{bmatrix}$$

The above stiffness matrix include the effects of axial force in the $\hat{x}$ direction, shear force in the $\hat{y}$, and bending moment about the $\hat{z}$ axis.

The local degrees of freedom may be related to the global degrees of freedom by:

$$\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1x} \\ \hat{\phi}_1 \\ \hat{d}_{2x} \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1x} \\ \phi_1 \\ d_{2x} \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

where the transformation matrix, including axial effects is:

$$\overline{T} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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Substituting the above transformation into the general form of the stiffness matrix $k = \overline{T}^T \hat{k} \overline{T}$ gives:

$$k = \frac{E}{L} \begin{bmatrix} AC^2 + \frac{12I}{L^2}S^2 & \left(A - \frac{12I}{L^2}\right)CS & -\frac{6I}{L}S & -\left(AC^2 + \frac{12I}{L^2}S^2\right) & -\left(A - \frac{12I}{L^2}\right)CS & -\frac{6I}{L}S \\ & AS^2 + \frac{12I}{L^2}C^2 & \frac{6I}{L}C & -\left(A - \frac{12I}{L^2}\right)CS & -\left(AS^2 + \frac{12I}{L^2}C^2\right) & \frac{6I}{L}C \\ & & 4I & \frac{6I}{L}S & -\frac{6I}{L}C & 2I \\ & & & AC^2 + \frac{12I}{L^2}S^2 & \left(A - \frac{12I}{L^2}\right)CS & \frac{6I}{L}S \\ & & & & AS^2 + \frac{12I}{L^2}C^2 & -\frac{6I}{L}C \\ \text{Symmetry} & & & & & 4I \end{bmatrix}$$

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•The analysis of a rigid plane frame can be undertaken by applying stiffness matrix.

•The element stiffnesses of a frame are functions of E,A,L,I, and the angle of orientation of the element with respect to the global-coordinate axes.

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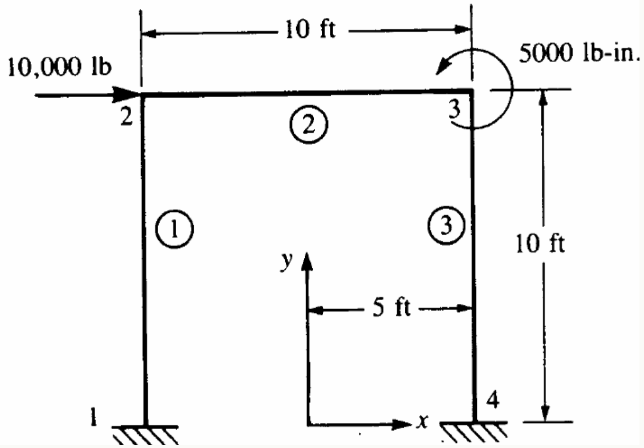
Rigid Plane Frame Example

Example 1

Consider the frame shown in the figure below.



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The frame is fixed at nodes 1 and 4 and subjected to a positive horizontal force of 10,000 lb applied at node 2 and to a positive moment of 5,000 lb-in. applied at node 3. Let E = 30 x 10^6 psi and A = 10 in.^{2} for all elements, and let I = 200 in.^{4} for elements 1 and 3, and I = 100 in.^{4} for element 2.

Element 1: The angle between x and $\hat{x}$ is 90°

$$C = 0 \qquad S = 1$$

where

$$\frac{12I}{L^2} = \frac{12(200)}{(120)^2} = 0.167\ in^2 \qquad \frac{6I}{L} = \frac{6(200)}{120} = 10.0\ in^3$$

$$\frac{E}{L} = \frac{30 \times 10^6}{120} = 250{,}000\ lb/in^3$$

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Therefore, for element 1:

$$k^{(1)} = 250{,}000 \begin{array}{c} \begin{matrix} d_{1x} & d_{1y} & \phi_1 & d_{2x} & d_{2y} & \phi_2 \end{matrix} \\ \begin{bmatrix} 0.167 & 0 & -10 & -0.167 & 0 & -10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ -10 & 0 & 800 & 10 & 0 & 400 \\ -0.167 & 0 & 10 & 0.167 & 0 & 10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ -10 & 0 & 400 & 10 & 0 & 800 \end{bmatrix} \end{array} lb/in$$

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Element 2: The angle between x and $\hat{x}$ is 0°

$$C = 1 \qquad S = 0$$

$$\frac{12I}{L^2} = \frac{12(100)}{(120)^2} = 0.0835\ in^2 \qquad \frac{6I}{L} = \frac{6(100)}{120} = 5.0\ in^3$$

Therefore, for element 2:

$$k^{(2)} = 250{,}000 \begin{array}{c} \begin{matrix} d_{2x} & d_{2y} & \phi_2 & d_{3x} & d_{3y} & \phi_3 \end{matrix} \\ \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0835 & 5 & 0 & 0.0835 & 5 \\ 0 & 5 & 400 & 0 & -5 & 200 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0.0835 & -5 & 0 & 0.0835 & -5 \\ 0 & 5 & 200 & 0 & -5 & 400 \end{bmatrix} \end{array} lb/in$$

Element 3: The angle between x and $\hat{x}$ is 270°

$$C = 0 \qquad S = -1$$

$$\frac{12I}{L^2} = \frac{12(200)}{(120)^2} = 0.167\ in^2 \qquad \frac{6I}{L} = \frac{6(200)}{120} = 10.0\ in^3$$

$$\frac{E}{L} = \frac{30\times 10^6}{120} = 250{,}000\ lb/in^3$$

Therefore, for element 3:

$$k^{(3)} = 250{,}000 \begin{array}{c} \begin{matrix} d_{3x} & d_{3y} & \phi_3 & d_{4x} & d_{4y} & \phi_4 \end{matrix} \\ \begin{bmatrix} 0.167 & 0 & 10 & -0.167 & 0 & 10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ 10 & 0 & 800 & -10 & 0 & 400 \\ -0.167 & 0 & -10 & 0.167 & 0 & -10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ 10 & 0 & 400 & -10 & 0 & 800 \end{bmatrix} \end{array} \ {}^{lb}\!/_{in}$$

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The boundary conditions for this problem are:

$$d_{1x} = d_{1y} = \phi_1 = d_{4x} = d_{4y} = \phi_4 = 0$$

After applying the boundary conditions the global beam equations reduce to:

$$\begin{Bmatrix} 10{,}000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5{,}000 \end{Bmatrix} = 2.5\times 10^5 \begin{bmatrix} 10.167 & 0 & 10 & -10 & 0 & 0 \\ 0 & 10.0835 & 5 & 0 & -0.0835 & 5 \\ 10 & 5 & 1200 & 0 & -5 & 200 \\ -10 & 0 & 0 & 10.167 & 0 & 10 \\ 0 & -0.0835 & -5 & 0 & 10.0835 & -5 \\ 0 & 5 & 200 & 10 & -5 & 1200 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{2y} \\ \phi_2 \\ d_{3x} \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

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Solving the above equations gives:

$$\begin{Bmatrix} d_{2x} \\ d_{2y} \\ \phi_2 \\ d_{3x} \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0.211\ in \\ 0.00148\ in \\ -0.00153\ rad \\ 0.209\ in \\ -0.00148\ in \\ -0.00149\ rad \end{Bmatrix}$$

Element 1: The element force-displacement equations can be obtained using $\hat{f} = \hat{k}\overline{T}d$. Therefore, $\overline{T}d$ is:

$$\overline{T}d = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} = 0 \\ d_{1y} = 0 \\ \phi_1 = 0 \\ d_{2x} = 0.211\ in \\ d_{2y} = 0.00148\ in \\ \phi_2 = -0.00153\ rad \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148\ in \\ -0.211\ in \\ -0.00153\ rad \end{Bmatrix}$$

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Recall the elemental stiffness matrix is:

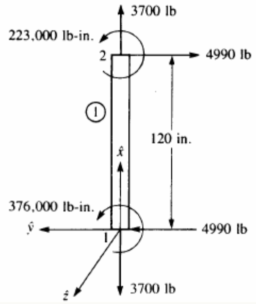
$$\hat{k} = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{bmatrix}$$

Therefore, the local force-displacement equations are:

$$\hat{f}^{(1)} = \hat{k}\overline{T}d = 2.5\times 10^5 \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.167 & 10 & 0 & -0.167 & 10 \\ 0 & 10 & 800 & 0 & -10 & 400 \\ -10 & 0 & 0 & 10 & 0 & 10 \\ 0 & -0.167 & -10 & 0 & 0.167 & -10 \\ 0 & 10 & 400 & 0 & -10 & 800 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148\ in \\ -0.211\ in \\ -0.00153\ rad \end{Bmatrix}$$

Simplifying the above equations gives:

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \begin{Bmatrix} -3{,}700\ lb \\ 4{,}990\ lb \\ 376\ k\cdot in \\ 3{,}700\ lb \\ -4{,}990\ lb \\ 223\ k\cdot in \end{Bmatrix}$$



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Element 2: The element force-displacement equations are:

$$\bar{T}d = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix} \begin{Bmatrix} d_{2x} = 0.211\ in \\ d_{2y} = 0.00148\ in \\ \phi_2 = -0.00153\ rad \\ d_{3x} = 0.209\ in \\ d_{3y} = -0.00148\ in \\ \phi_3 = -0.00149\ rad \end{Bmatrix} = \begin{Bmatrix} -0.211\ in \\ 0.00148\ in \\ -0.00153\ rad \\ 0.209\ in \\ -0.00148\ in \\ -0.00149\ rad \end{Bmatrix}$$

Therefore, the local force-displacement equations are:

$$\hat{f}^{(2)} = \hat{k}\bar{T}d = 2.5\times 10^5 \begin{bmatrix} 10&0&0&-10&0&0 \\ 0&0.0833&5&0&-0.0833&5 \\ 0&5&400&0&-5&200 \\ -10&0&0&10&0&0 \\ 0&-0.0833&-5&0&0.0833&-5 \\ 0&5&200&0&-5&400 \end{bmatrix} \begin{Bmatrix} 0.211\ in \\ 0.00148\ in \\ -0.00153\ rad \\ 0.209\ in \\ -0.00148\ in \\ -0.00149\ rad \end{Bmatrix}$$

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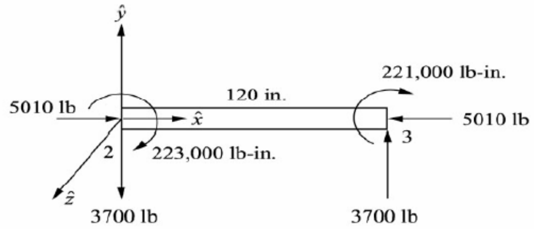
Simplifying the above equations gives:

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \\ \hat{f}_{3x} \\ \hat{f}_{3y} \\ \hat{m}_3 \end{Bmatrix} = \begin{Bmatrix} 5{,}010\ lb \\ -3{,}700\ lb \\ -223\ k\cdot in \\ -5{,}010\ lb \\ 3{,}700\ lb \\ -221\ k\cdot in \end{Bmatrix}$$



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Element 3: The element force-displacement equations are:

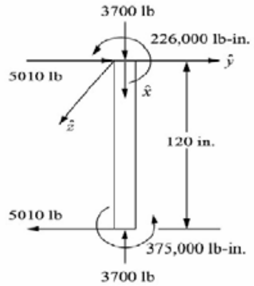
$$\bar{T}d = \begin{bmatrix} 0&-1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&-1&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \end{bmatrix} \begin{Bmatrix} d_{3x} = 0.209\ in \\ d_{3y} = -0.00148\ in \\ \phi_3 = -0.00149\ rad \\ d_{4x} = 0 \\ d_{4y} = 0 \\ \phi_4 = 0 \end{Bmatrix} = \begin{Bmatrix} 0.00148\ in \\ 0.209\ in \\ -0.00149\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Therefore, the local force-displacement equations are:

$$\hat{f}^{(3)} = \hat{k}\bar{T}d = 2.5\times10^{5}\begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.167 & 10 & 0 & -0.167 & 10 \\ 0 & 10 & 800 & 0 & -10 & 400 \\ -10 & 0 & 0 & 10 & 0 & 10 \\ 0 & -0.167 & -10 & 0 & 0.167 & -10 \\ 0 & 10 & 400 & 0 & -10 & 800 \end{bmatrix}\begin{Bmatrix} 0.00148\ in \\ 0.209\ in \\ -0.00149\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Simplifying the above equations gives:

$$\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{3y} \\ \hat{m}_{3} \\ \hat{f}_{4x} \\ \hat{f}_{4y} \\ \hat{m}_{4} \end{Bmatrix} = \begin{Bmatrix} 3{,}700\ lb \\ 5{,}010\ lb \\ 226\ k\cdot in \\ -3{,}700\ lb \\ -5{,}010\ lb \\ 375\ k\cdot in \end{Bmatrix}$$



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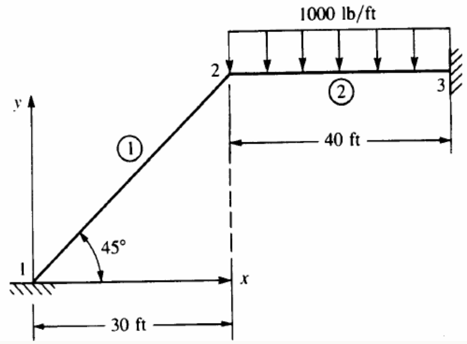
Example 2

Consider the frame shown in the figure below.



The frame is fixed at nodes 1 and 3 and subjected to a positive distributed load of 1,000 lb/ft applied along element 2. Let E = 30 x 10^6 psi and A = 100 in.^{2} for all elements, and let I = 1,000 in.^{4} for all elements.

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First we need to replace the distributed load with a set of equivalent nodal forces and moments acting at nodes 2 and 3. For a beam with both end fixed, subjected to a uniform distributed load, w, the nodal forces and moments are:

$$f_{2y} = f_{3y} = -\frac{wL}{2} = -\frac{(1{,}000\ lb/ft)40\ ft}{2} = -20k$$

$$m_2 = -m_3 = -\frac{wL^2}{12} = -\frac{(1{,}000\ lb/ft)(40\ ft)^2}{12} = -133{,}333\ lb\cdot ft = 1{,}600\ k\cdot in$$

If we consider only the parts of the stiffness matrix associated with the three degrees of freedom at node 2, we get:

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Element 1: The angle between x and $\hat{x}$ is 45º

$$C = 0.707 \qquad S = 0.707$$

where

$$\frac{E}{L} = \frac{30\times10^6}{509} = 58.93\ k/in^3 \qquad \frac{12I}{L^2} = \frac{12(1{,}000)}{\left(12\times30\sqrt{2}\right)^2} = 0.0463\ in^2$$

$$\frac{6I}{L} = \frac{6(1{,}000)}{12\times30\sqrt{2}} = 11.78551\ in^3$$

Therefore, for element 1:

$$k^{(1)} = 58.93\begin{bmatrix} 50.02 & 49.98 & 8.33 \\ 49.98 & 50.02 & -8.33 \\ 8.33 & -8.33 & 4000 \end{bmatrix} k/in \quad (d_{2x},\ d_{2y},\ \phi_2)$$

Simplifying the above equation:

$$k^{(1)} = \begin{bmatrix} 2{,}948 & 2{,}945 & 491 \\ 2{,}945 & 2{,}948 & -491 \\ 491 & -491 & 235{,}700 \end{bmatrix} k/in \quad (d_{2x},\ d_{2y},\ \phi_2)$$

Element 2: The angle between x and x is 0°

$$C = 1 \qquad S = 0$$

where

$$\frac{E}{L} = \frac{30\times10^6}{480} = 62.5\ k/in^3 \qquad \frac{12I}{L^2} = \frac{12(1{,}000)}{(12\times40)^2} = 0.0521\ in^2$$

$$\frac{6I}{L} = \frac{6(1{,}000)}{12\times40} = 12.5\ in^3$$

Therefore, for element 2:

$$k^{(2)} = 62.50\begin{matrix} d_{2x} & d_{2y} & \phi_2 \\ \end{matrix}\begin{bmatrix} 100 & 0 & 0 \\ 0 & 0.052 & 12.5 \\ 0 & 12.5 & 4{,}000 \end{bmatrix} {}^{k}\!/_{in}$$

Simplifying the above equation:

$$k^{(2)} = \begin{matrix} d_{2x} & d_{2y} & \phi_2 \\ \end{matrix}\begin{bmatrix} 6{,}250 & 0 & 0 \\ 0 & 3.25 & 781.25 \\ 0 & 781.25 & 250{,}000 \end{bmatrix} {}^{k}\!/_{in}$$

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The global beam equations reduce to:

$$\begin{Bmatrix} 0 \\ -20\ k \\ -1{,}600\ k\cdot in \end{Bmatrix} = \begin{bmatrix} 9{,}198 & 2{,}945 & 491 \\ 2{,}945 & 2{,}951 & 290 \\ 491 & 290 & 485{,}700 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} d_{2x} \\ d_{2y} \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \end{Bmatrix}$$

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Element 1: The element force-displacement equations can be obtained using $\hat{f} = \hat{k}\overline{T}d$. Therefore, $\overline{T}d$ is:

$$\overline{T}d = \begin{bmatrix} 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ -0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.707 & 0.707 & 0 \\ 0 & 0 & 0 & -0.707 & 0.707 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00452\ in \\ -0.0092\ in \\ -0.0033\ rad \end{Bmatrix}$$

Recall the elemental stiffness matrix is a function of values C_1, C_2, and L

$$C_1 = \frac{AE}{L} = \frac{(100)30\times10^6}{12\times30\sqrt{2}} = 5{,}893\ {}^{k}\!/_{in} \qquad C_2 = \frac{EI}{L^3} = \frac{30\times10^6(1{,}000)}{\left(12\times30\sqrt{2}\right)^3} = 0.2273\ {}^{k}\!/_{in}$$

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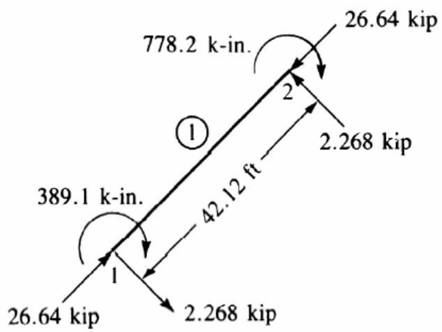
Therefore, the local force-displacement equations are:

$$\hat{f}_{(1)} = \hat{k}\overline{T}d = \begin{bmatrix} 5{,}893 & 0 & 10 & -5{,}893 & 0 & 0 \\ 0 & 2.730 & 694.8 & 0 & -2.730 & 694.8 \\ 10 & 694.8 & 117{,}900 & 0 & -694.8 & 117{,}000 \\ -5{,}893 & 0 & 0 & 5{,}983 & 0 & 0 \\ 0 & -2.730 & -694.8 & 0 & 2.730 & -694.8 \\ 0 & 694.8 & 117{,}000 & 0 & -694.8 & 235{,}800 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00452\ in \\ -0.0092\ in \\ -0.0033\ rad \end{Bmatrix}$$

Simplifying the above equations gives:

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \begin{Bmatrix} 26.64\ k \\ -2.268\ k \\ -389.1\ k \cdot in \\ -26.64\ k \\ 2.268\ k \\ -778.2\ k \cdot in \end{Bmatrix}$$



Element 2: The element force-displacement equations are:

$$\bar{T}d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Recall the elemental stiffness matrix is a function of values C_1, C_2, and L

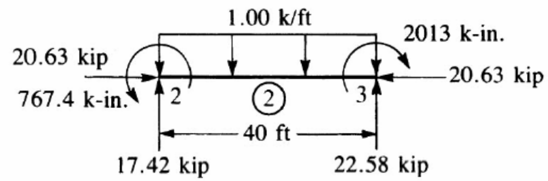
$$C_1 = \frac{AE}{L} = \frac{(100)30 \times 10^6}{12 \times 40} = 6{,}250\ {}^{k}/_{in} \qquad C_2 = \frac{EI}{L^3} = \frac{30 \times 10^6 (1{,}000)}{(12 \times 40)^3} = 0.2713\ {}^{k}/_{in}$$

Therefore, the local force-displacement equations are:

$$\hat{f}_{(2)} = \hat{k}\bar{T}d = \begin{bmatrix} 6{,}250 & 0 & 0 & -6{,}250 & 0 & 0 \\ 0 & 3.25 & 781.1 & 0 & -3.25 & 781.1 \\ 0 & 781.1 & 250{,}000 & 0 & -781.1 & 125{,}000 \\ -6{,}250 & 0 & 0 & 6{,}250 & 0 & 0 \\ 0 & -3.25 & -781.1 & 0 & 3.25 & -781.1 \\ 0 & 781.1 & 125{,}000 & 0 & -781.1 & 250{,}00 \end{bmatrix} \begin{Bmatrix} -0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Simplifying the above equations gives:

$$\hat{k}\hat{d} = \begin{Bmatrix} 20.63\ k \\ -2.58\ k \\ -832.57\ k \cdot in \\ -20.63\ k \\ 2.58\ k \\ -412.50\ k \cdot in \end{Bmatrix}$$



To obtain the actual element local forces, we must subtract the equivalent nodal forces.

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \\ \hat{f}_{3x} \\ \hat{f}_{3y} \\ \hat{m}_3 \end{Bmatrix} = \begin{Bmatrix} 20.63\ k \\ -2.58\ k \\ -832.57\ k\cdot in \\ -20.63\ k \\ 2.58\ k \\ -412.50\ k\cdot in \end{Bmatrix} - \begin{Bmatrix} 0 \\ -20\ k \\ -1600\ k\cdot in \\ 0 \\ -20\ k \\ 1600\ k\cdot in \end{Bmatrix} = \begin{Bmatrix} 20.63\ k \\ 17.42\ k \\ 767.4\ k\cdot in \\ -20.63\ k \\ 22.58\ k \\ -2{,}013\ k\cdot in \end{Bmatrix}$$

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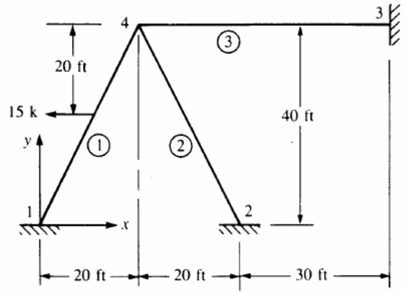
Example 3



The frame is fixed at nodes 1, 2, and 3 and subjected to a concentrated load of 15 *k* applied at mid-length of element 1. Let $E = 30 \times 10^6$ psi, $A = 8$ in^2, and let $I = 800$ in^4 for all elements.

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Consider the frame shown in the figure . In this example will illustrate the equivalent joint force replacement method for a frame subjected to a load acting on an element instead of at one of the joints of the structure. Since no distributed loads are present, the point of application of the concentrated load could be treated as an extra joint in the analysis.

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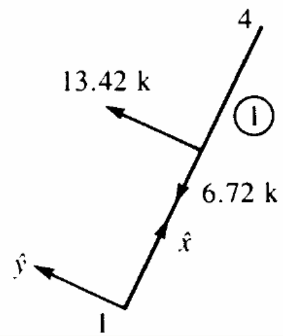
Solution Procedure

1. Express the applied load in the element 1 local coordinate system (here $\hat{x}$ is directed from node 1 to node 4).



2. Next, determine the equivalent joint forces at each end of element 1, using the table in Appendix D (see figure below).

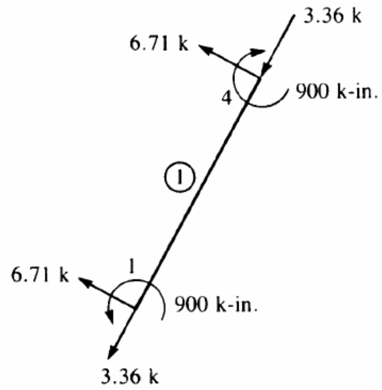


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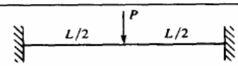
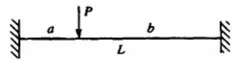
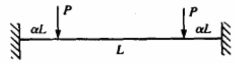
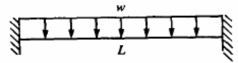
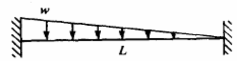
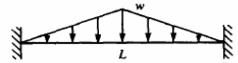
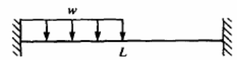
Table D–1 Equivalent joint forces f_0 for different types of loads

	f_{1y}	m_1	Loading case	f_{2y}	m_2
1.	$\frac{-P}{2}$	$\frac{-PL}{8}$		$\frac{-P}{2}$	$\frac{PL}{8}$
2.	$\frac{-Pb^2(L+2a)}{L^3}$	$\frac{-Pab^2}{L^2}$		$\frac{-Pa^2(L+2b)}{L^3}$	$\frac{Pa^2b}{L^2}$
3.	$-P$	$-\alpha(1-\alpha)PL$		$-P$	$\alpha(1-\alpha)PL$
4.	$\frac{-wL}{2}$	$\frac{-wL^2}{12}$		$\frac{-wL}{2}$	$\frac{wL^2}{12}$
5.	$\frac{-7wL}{20}$	$\frac{-wL^2}{20}$		$\frac{-3wL}{20}$	$\frac{wL^2}{30}$
6.	$\frac{-wL}{4}$	$\frac{-5wL^2}{96}$		$\frac{-wL}{4}$	$\frac{5wL^2}{96}$
7.	$\frac{-13wL}{32}$	$\frac{-11wL^2}{192}$		$\frac{-3wL}{32}$	$\frac{5wL^2}{192}$

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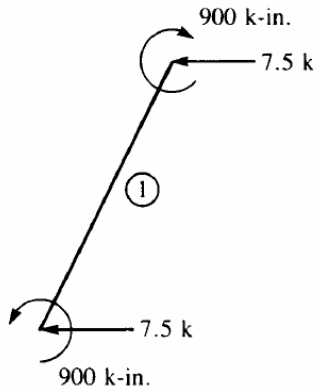
3. Then transform the equivalent joint forces from the local coordinate system forces into the global coordinate system forces, using the equation $f = T^T\hat{f}$. These global joint forces are shown below.



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4. Then we analyze the structure, using the equivalent joint forces (plus actual joint forces, if any) in the usual manner.

5. The final internal forces developed at the ends of each element may be obtained by subtracting Step 2 joint forces from Step 4 joint forces.

Element 1: The angle between x and $\hat{x}$ is 63.43°

$C = 0.447 \qquad S = 0.895$

where

$$\frac{12I}{L^2} = \frac{12(800)}{(44.7 \times 12)^2} = 0.0334\ in^2 \qquad \frac{6I}{L} = \frac{6(800)}{44.7 \times 12} = 8.95\ in^3$$

$$\frac{E}{L} = \frac{30 \times 10^6}{44.7 \times 12} = 55.9\ k/in^3$$

Therefore, for element 1:

$$k^{(1)} = \begin{bmatrix} 90.0 & 178 & 448 \\ 178 & 359 & -244 \\ 448 & -244 & 179{,}000 \end{bmatrix} \ {}^{k}/_{in}$$

(columns: d_{4x}, d_{4y}, ϕ_4)

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Element 2: The angle between x and $\hat{x}$ is 116.57°

$C = -0.447 \qquad S = 0.895$

where

$$\frac{12I}{L^2} = \frac{12(800)}{(44.7 \times 12)^2} = 0.0334\ in^2 \qquad \frac{6I}{L} = \frac{6(800)}{44.7 \times 12} = 8.95\ in^3$$

$$\frac{E}{L} = \frac{30 \times 10^6}{44.7 \times 12} = 55.9\ k/in^3$$

Therefore, for element 2:

$$k^{(2)} = \begin{bmatrix} 90.0 & -178 & 448 \\ -178 & 359 & 244 \\ 448 & 244 & 179{,}000 \end{bmatrix} \ {}^{k}/_{in}$$

(columns: d_{4x}, d_{4y}, ϕ_4)

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Element 3: The angle between x and $\hat{x}$ is 0° (The author of your textbook directed the element from node 4 to 3. In general, as we have discussed in class, we usually number the element numerically or from 3 to 4. In this case the angle between x and $\hat{x}$ is 180°)

$C = 1 \qquad S = 0 \qquad \frac{E}{L} = \frac{30 \times 10^6}{50 \times 12} = 50\ k/in^3$

$$\frac{12I}{L^2} = \frac{12(800)}{(50 \times 12)^2} = 0.0267\ in^2 \qquad \frac{6I}{L} = \frac{6(800)}{50 \times 12} = 8.0\ in^3$$

Therefore, for element 3:

$$k^{(2)} = \begin{bmatrix} 400 & 0 & 0 \\ 0 & 1.334 & 400 \\ 0 & 400 & 160{,}000 \end{bmatrix} \ {}^{k}/_{in}$$

(columns: d_{4x}, d_{4y}, ϕ_4)

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The global beam equations reduce to:

$$\begin{Bmatrix} -7.5\ k \\ 0 \\ -900\ k \cdot in \end{Bmatrix} = \begin{bmatrix} 582 & 0 & 896 \\ 0 & 719 & 400 \\ 896 & 400 & 518{,}000 \end{bmatrix} \begin{Bmatrix} d_{4x} \\ d_{4y} \\ \phi_4 \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} d_{4x} \\ d_{4y} \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} -0.0103\ in \\ 0.000956\ in \\ -0.00172\ rad \end{Bmatrix}$$

Element 1: The element force-displacement equations can be obtained using $\hat{f} = \hat{k}\overline{T}d$. Therefore, $\overline{T}d$ is:

$$\overline{T} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad C = 0.447 \qquad S = 0.895$$

$$\overline{T}d = \begin{bmatrix} 0.447 & 0.895 & 0 & 0 & 0 & 0 \\ -0.895 & 0.447 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.447 & 0.895 & 0 \\ 0 & 0 & 0 & -0.895 & 0.447 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0103\ in \\ 0.000956\ in \\ -0.00172\ rad \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00374\ in \\ 0.00963\ in \\ -0.00172\ rad \end{Bmatrix}$$

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Recall the elemental stiffness matrix is:

$$\hat{k} = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{bmatrix}$$

$$C_1 = \frac{AE}{L} = \frac{(8)30 \times 10^6}{12 \times 44.72} = 447.2\ {}^{k}/_{in} \qquad C_2 = \frac{EI}{L^3} = \frac{30 \times 10^6 (800)}{(12 \times 44.72)^3} = 0.155\ {}^{k}/_{in}$$

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Therefore, the local force-displacement equations are:

$$\hat{f}_{(1)} = \hat{k}\overline{T}d = \begin{bmatrix} 447 & 0 & 0 & -447 & 0 & 0 \\ 0 & 1.868 & 500.5 & 0 & -1.868 & 500.5 \\ 0 & 500.5 & 179{,}000 & 0 & -500.5 & 89{,}490 \\ -447 & 0 & 0 & 447 & 0 & 0 \\ 0 & -1.868 & -500.5 & 0 & 1.868 & -500.5 \\ 0 & 500.5 & 89{,}490 & 0 & -500.5 & 179{,}000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00374\ in \\ 0.00963\ in \\ -0.00172\ rad \end{Bmatrix}$$

Simplifying the above equations gives:

$$\hat{f}_{(1)} = \hat{k}\hat{d} = \begin{Bmatrix} 1.67\ k \\ -0.88\ k \\ -158\ k \cdot in \\ -1.67\ k \\ 0.88\ k \\ -311\ k \cdot in \end{Bmatrix}$$

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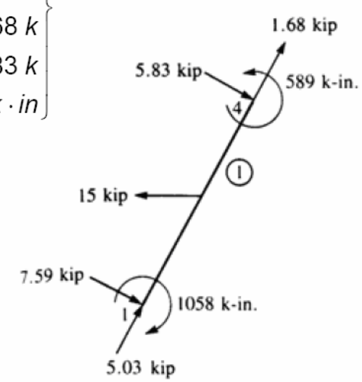
To obtain the actual element local forces, we must subtract the equivalent nodal forces.

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{4x} \\ \hat{f}_{4y} \\ \hat{m}_4 \end{Bmatrix} = \begin{Bmatrix} 1.67\ k \\ -0.88\ k \\ -158\ k \cdot in \\ -1.67\ k \\ 0.88\ k \\ -311\ k \cdot in \end{Bmatrix} - \begin{Bmatrix} -3.36\ k \\ 6.71\ k \\ 900\ k \cdot in \\ -3.36\ k \\ 6.71\ k \\ -900\ k \cdot in \end{Bmatrix} = \begin{Bmatrix} 5.03\ k \\ -7.59\ k \\ -1{,}058\ k \cdot in \\ 1.68\ k \\ -5.83\ k \\ 589\ k \cdot in \end{Bmatrix}$$



Element 2: The element force-displacement equations can be obtained using $\hat{f} = \hat{k}\overline{T}d$. Therefore, $\overline{T}d$ is:

$$\overline{T} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad C = -0.447 \qquad S = 0.895$$

$$\overline{T}d = \begin{bmatrix} -0.447 & 0.895 & 0 & 0 & 0 & 0 \\ -0.895 & -0.447 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.447 & 0.895 & 0 \\ 0 & 0 & 0 & -0.895 & -0.447 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0103\ in \\ 0.000956\ in \\ -0.00172\ rad \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00546\ in \\ 0.00879\ in \\ -0.00172\ rad \end{Bmatrix}$$

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Therefore, the local force-displacement equations are:

$$C_1 = \frac{AE}{L} = \frac{(8)30 \times 10^6}{12 \times 44.72} = 447.2\ {}^{k}\!/_{in} \qquad C_2 = \frac{EI}{L^3} = \frac{30 \times 10^6 (800)}{(12 \times 44.72)^3} = 0.155\ {}^{k}\!/_{in}$$

$$\hat{f}_{(2)} = \hat{k}\overline{T}d = \begin{bmatrix} 447 & 0 & 0 & -447 & 0 & 0 \\ 0 & 1.868 & 500.5 & 0 & -1.868 & 500.5 \\ 0 & 500.5 & 179{,}000 & 0 & -500.5 & 89{,}490 \\ -447 & 0 & 0 & 447 & 0 & 0 \\ 0 & -1.868 & -500.5 & 0 & 1.868 & -500.5 \\ 0 & 500.5 & 89{,}490 & 0 & -500.5 & 179{,}000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00546\ in \\ 0.00879\ in \\ -0.00172\ rad \end{Bmatrix}$$

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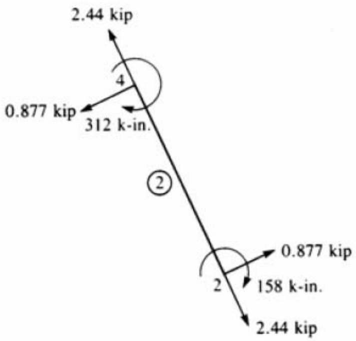
Simplifying the above equations gives:

$$\hat{f}_{(2)} = \hat{k}\hat{d} = \begin{Bmatrix} 2.44\ k \\ -0.877\ k \\ -158\ k \cdot in \\ 2.44\ k \\ 0.877\ k \\ -312\ k \cdot in \end{Bmatrix}$$



Since there are no applied loads on element 2, there are no equivalent nodal forces to account for. Therefore, the above equations are the final local nodal forces

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Element 3: The element force-displacement equations can be obtained using $\hat{f} = \hat{k}\overline{T}d$. Therefore, $\overline{T}d$ is:

$$\overline{T}d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -0.0103\ in \\ 0.000956\ in \\ -0.00172\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.0103\ in \\ 0.000956\ in \\ -0.00172\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

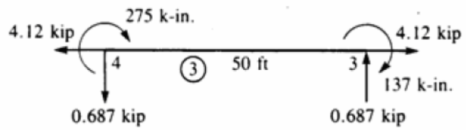
Therefore, the local force-displacement equations are:

$$C_1 = \frac{AE}{L} = \frac{(8)30 \times 10^6}{12 \times 50} = 400\ {}^{k}\!/_{in} \qquad C_2 = \frac{EI}{L^3} = \frac{30 \times 10^6 (800)}{(12 \times 50)^3} = 0.111\ {}^{k}\!/_{in}$$

$$\hat{f}_{(3)} = \hat{k}\bar{T}d = \begin{bmatrix} 400 & 0 & 0 & -400 & 0 & 0 \\ 0 & 1.335 & 400 & 0 & -1.335 & 400 \\ 0 & 400 & 160{,}000 & 0 & -400 & 80{,}000 \\ -400 & 0 & 0 & 400 & 0 & 0 \\ 0 & -1.335 & -400 & 0 & 1.335 & -400 \\ 0 & 400 & 80{,}000 & 0 & -400 & 160{,}000 \end{bmatrix} \begin{Bmatrix} -0.0103\ in \\ 0.000956\ in \\ -0.00172\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Simplifying the above equations gives:

$$\hat{f}_{(3)} = \hat{k}\hat{d} = \begin{Bmatrix} -4.12\ k \\ -0.687\ k \\ -275\ k \cdot in \\ 4.12\ k \\ 0.687\ k \\ -137\ k \cdot in \end{Bmatrix}$$



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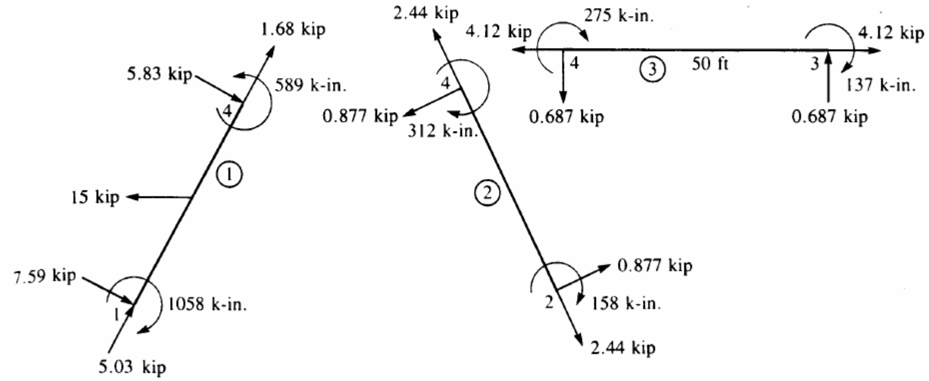
Since there are no applied loads on element 3, there are no equivalent nodal forces to account for. Therefore, the above equations are the final local nodal forces. The free-body diagrams are shown below.



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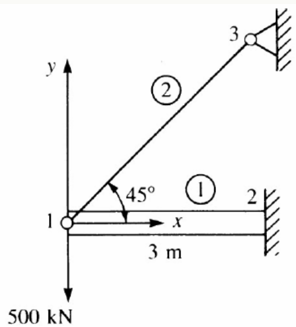
Example 4

The frame shown on the right is fixed at nodes 2 and 3 and subjected to a concentrated load of 500 *kN* applied at node 1. For the bar, A = 1 x 10^{-3} m^2, for the beam, A = 2 x 10^{-3} m^2, I = 5 x 10^{-5} m^4, and L = 3 m. Let E = 210 GPa for both elements.



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Beam Element 1: The angle between x and $\hat{x}$ is 0°

$C = 1 \qquad S = 0$

where

$$\frac{12I}{L^2} = \frac{12(5\times10^{-5})}{(3)^2} = 6.67\times10^{-5}\ m^2 \qquad \frac{6I}{L} = \frac{6(5\times10^{-5})}{3} = 10^{-4}\ m^3$$

$$\frac{E}{L} = \frac{210\times10^6}{3} = 70\times10^6\ kN/m^3$$

Therefore, for element 1:

$$k^{(1)} = 70\times10^3 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.067 & 0.10 \\ 0 & 0.10 & 0.20 \end{bmatrix} {kN}/{m}$$

(columns: d_{1x}, d_{1y}, ϕ_1)

Bar Element 2: The angle between x and $\hat{x}$ is 45°

$C = 0.707 \qquad S = 0.707$

where

$$k^{(2)} = \frac{10^{-3}\,m^2(210\times10^6\,kN/m^2)}{4.24\,m}\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} kN/m \quad (d_{1x},\ d_{1y})$$

$$k^{(2)} = 70\times10^3\begin{bmatrix} 0.354 & 0.354 \\ 0.354 & 0.354 \end{bmatrix} kN/m \quad (d_{1x},\ d_{1y})$$

Assembling the elemental stiffness matrices we obtain the global stiffness matrix

$$K = 70\times10^3\begin{bmatrix} 2.354 & 0.354 & 0 \\ 0.354 & 0.421 & 0.10 \\ 0 & 0.10 & 0.20 \end{bmatrix} kN/m$$

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The global equations are:

$$\begin{Bmatrix} 0 \\ -500\ kN \\ 0 \end{Bmatrix} = 70\times10^3\ kN/m \begin{bmatrix} 2.354 & 0.354 & 0 \\ 0.354 & 0.421 & 0.10 \\ 0 & 0.10 & 0.20 \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ \phi_1 \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} d_{1x} \\ d_{1y} \\ \phi_1 \end{Bmatrix} = \begin{Bmatrix} 0.00388\ m \\ -0.0225\ m \\ 0.0113\ rad \end{Bmatrix}$$

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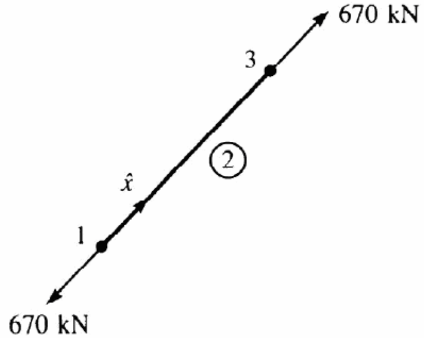
Bar Element: The bar element force-displacement equations can be obtained using $\hat{f} = \hat{k}T d$.

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{3x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$

Therefore, the forces in the bar element are:

$$\hat{f}_{1x} = \frac{AE}{L}(Cd_{1x} + Sd_{1y}) = -670\ kN$$

$$\hat{f}_{3x} = -\frac{AE}{L}(Cd_{1x} + Sd_{1y}) = 670\ kN$$



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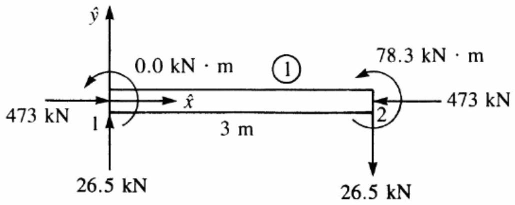
Beam Element: The beam element force-displacement equations can be obtained using $\hat{f} = \hat{k}\hat{d}$. Since the local axis coincides with the global coordinate system, and the displacements at node 2 are zero. Therefore, the local force-displacement equations are:

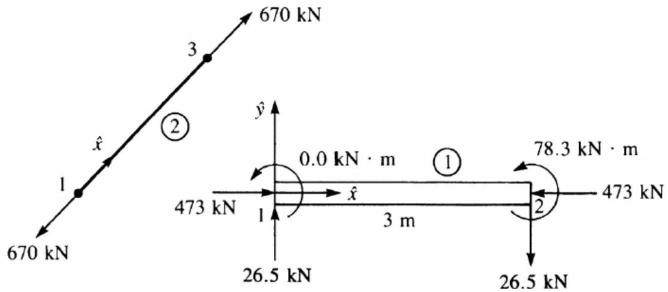
$$\hat{k} = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{bmatrix} \qquad C_1 = \frac{AE}{L} \qquad C_2 = \frac{EI}{L^3}$$

$$\hat{f}_{(1)} = \hat{k}\hat{d} = 70\times10^3\begin{bmatrix} 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0.067 & 0.10 & 0 & -0.067 & 0.10 \\ 0 & 0.10 & 0.20 & 0 & -0.10 & 0.10 \\ -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & -0.067 & -0.10 & 0 & 0.067 & -0.10 \\ 0 & 0.10 & 0.10 & 0 & -0.10 & 0.20 \end{bmatrix}\begin{Bmatrix} 0.00388\ m \\ -0.0225\ m \\ 0.0113\ kN\cdot m \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Substituting numerical values into the above equations gives:

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \begin{Bmatrix} 473\ kN \\ -26.5\ kN \\ 0.0 \\ -473\ kN \\ 26.5\ kN \\ -78.3\ kN \cdot m \end{Bmatrix}$$

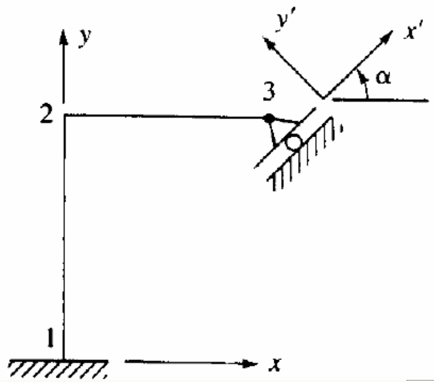




Inclined or Skewed Supports
Frame Example Problems

Example 5

If a support is inclined, or skewed, at some angle α for the global *x* axis, as shown below, the boundary conditions on the displacements are not in the global *x-y* directions but in the *x'-y'* directions.



We must transform the local boundary condition of $d'_{3y} = 0$ (in local coordinates) into the global *x-y* system. Therefore, the relationship between of the components of the displacement in the local and the global coordinate systems at node 3 is:

$$\begin{Bmatrix} d'_{3x} \\ d'_{3y} \\ \phi'_3 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

We can rewrite the above expression as:

$$\{d'_3\} = [t_3]\{d_3\} \qquad [t_3] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can apply this sort of transformation to the entire displacement vector as:

$$\{d'\} = [T_i]\{d\} \qquad \text{or} \qquad \{d\} = [T_i]^T\{d'\}$$

where the matrix $[T_i]$ is:

$$[T_i] = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3] \end{bmatrix}$$

Both the identity matrix **[*I*]** and the matrix **[*t*$_3$]** are 3 x 3 matrices.

The force vector can be transformed by using the same transformation.

$$\{f'\} = [T_i]\{f\}$$

In global coordinates, the force-displacement equations are:

$$\{f\} = [K]\{d\}$$

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Applying the skewed support transformation to both sides of the force-displacement equation gives:

$$[T_i]\{f\} = [T_i][K]\{d\}$$

By using the relationship between the local and the global displacements, the force-displacement equations become:

$$[T_i]\{f\} = [T_i][K][T_i]^T\{d'\} \quad \Rightarrow \quad \{f'\} = [T_i][K][T_i]^T\{d'\}$$

Therefore the global equations become:

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \\ F'_{3x} \\ F'_{3y} \\ M_3 \end{Bmatrix} = [T_i][K][T_i]^T \begin{Bmatrix} d_{1x} \\ d_{1y} \\ \phi_1 \\ d_{2x} \\ d_{2y} \\ \phi_2 \\ d'_{3x} \\ d'_{3y} \\ \phi_1 \end{Bmatrix}$$

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Development of Grid Equations

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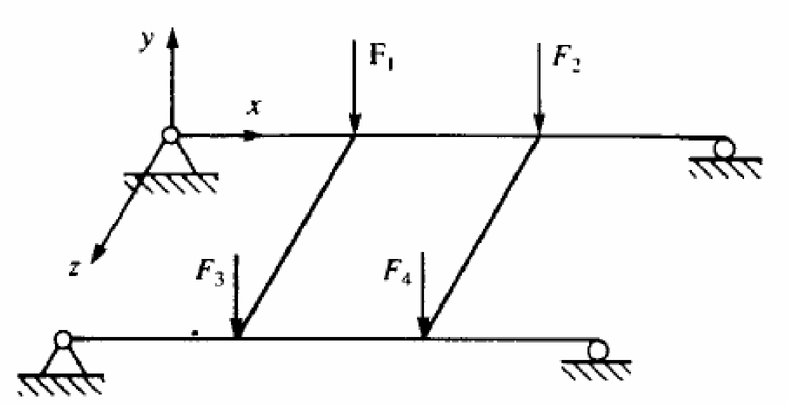
Grid Structures

- A grid is a structure on which the loads are applied perpendicular to the plane of the structure as opposed to a plane frame where loads are applied in the plane.
- Both torsional and bending moment continuity are maintained at each node in a grid element.
- Examples are floors and bridge deck systems.

A typical grid structure is shown in the figure below.

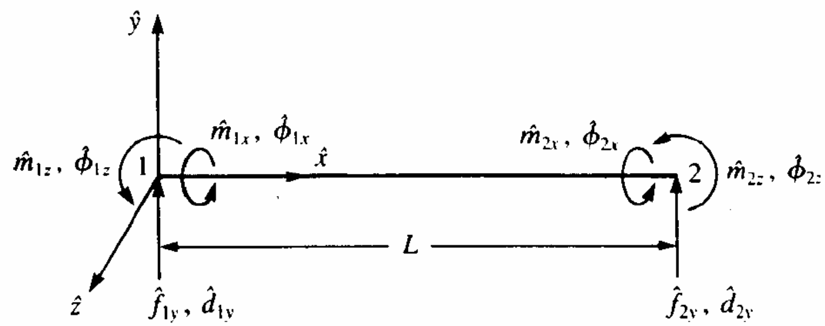


• Direct Stiffness – grid

Summary:

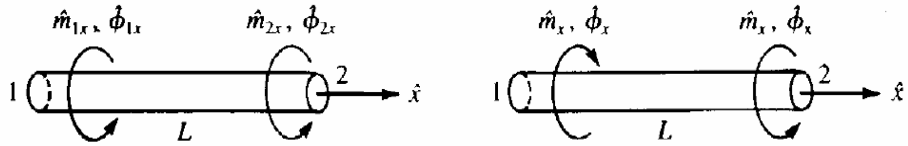
- **Derivation of the torsional components of the element matrix .**
- **Local stiffness matrix of a beam element oriented in space.**
- **Example Problems**

A representation of the grid element is shown below:



The degrees of freedom for a grid element are: a vertical displacement $\hat{d}_{iy}$ (normal to the grid), a torsional rotation $\hat{\phi}_{ix}$ about the $\hat{x}$ axis, and a bending rotation $\hat{\phi}_{iz}$ about the $\hat{z}$ axis. The nodal forces are: a transverse force $\hat{f}_{iy}$ a torsional moment $\hat{m}_{ix}$ about the $\hat{x}$ axis, and a bending moment $\hat{m}_{iz}$ about the $\hat{z}$ axis.

Let's derive the torsional rotation components of the element stiffness matrix. Consider the sign convention for nodal torque and angle of twist shown the figure below.



A linear displacement function $\hat{\phi}$ is assumed.

$$\phi = a_1 + a_2\hat{x}$$

Or in matrix form:

$$\hat{\phi} = [N_1 \quad N_2] = \begin{Bmatrix} \hat{\phi}_{1x} \\ \hat{\phi}_{2x} \end{Bmatrix}$$

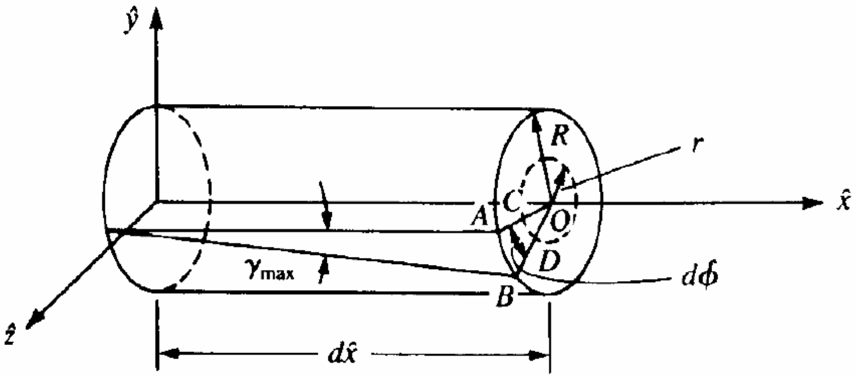
Applying the boundary conditions and solving for the unknown coefficients gives:

$$\phi = \left(\frac{\hat{\phi}_{2x} - \hat{\phi}_{1x}}{L}\right)\hat{x} + \hat{\phi}_{1x}$$

$$N_1 = 1 - \frac{\hat{x}}{L} \qquad N_2 = \frac{x}{L}$$

N_1* and *N_2 are the interpolation functions

To obtain the relationship between the shear strain γ and the angle of twist $\hat{\phi}$ consider the torsional deformation of the bar as shown below.



If we assume that all radial lines, such as ***OA***, remain straight during twisting or torsional deformation, then the arc length $\overline{AB}$ is:

$$\overline{AB} = \gamma_{max} d\hat{x} = R d\hat{\phi}$$

Therefore;

$$\gamma_{max} = \frac{R d\hat{\phi}}{d\hat{x}}$$

At any radial position, *r*, we have, from similar triangles ***OAB*** and ***OCD***:

$$\gamma = r\frac{d\hat{\phi}}{d\hat{x}} = \frac{r}{L}\left(\hat{\phi}_{2x} - \hat{\phi}_{1x}\right)$$

The relationship between shear stress and shear strain is:

$\tau = G\gamma$ where ***G*** is the ***shear modulus*** of the material.

From elementary mechanics of materials, we get:

$$\hat{m}_x = \frac{\tau J}{R}$$

where ***J*** is the ***polar moment of inertia*** for a circular cross section or the ***torsional constant*** for non-circular cross sections. Rewriting the above equation we get:

$$\hat{m}_x = \frac{GJ}{L}\left(\hat{\phi}_{2x} - \hat{\phi}_{1x}\right)$$

The nodal torque sign convention gives:

$$\hat{m}_{1x} = -\hat{m}_x \qquad \hat{m}_{2x} = \hat{m}_x$$

Therefore;

$$\hat{m}_{1x} = \frac{GJ}{L}\left(\hat{\phi}_{1x} - \hat{\phi}_{2x}\right) \qquad \hat{m}_{2x} = \frac{GJ}{L}\left(\hat{\phi}_{2x} - \hat{\phi}_{1x}\right)$$

In matrix form

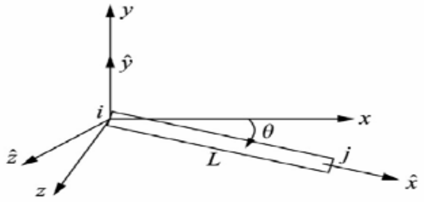
$$\begin{Bmatrix} \hat{m}_{1x} \\ \hat{m}_{2x} \end{Bmatrix} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{\phi}_{1x} \\ \hat{\phi}_{2x} \end{Bmatrix}$$

Combining the torsional effects with shear and bending effects, we obtain the local stiffness matrix equations for a grid element.

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_{1x} \\ \hat{m}_{1z} \\ \hat{f}_{2y} \\ \hat{m}_{2x} \\ \hat{m}_{2z} \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} \\ 0 & \frac{GJ}{L} & 0 & 0 & -\frac{GJ}{L} & 0 \\ \frac{6EI}{L^2} & 0 & \frac{4EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\ 0 & -\frac{GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 \\ \frac{6EI}{L^2} & 0 & \frac{2EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{4EI}{L} \end{bmatrix}\begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_{1x} \\ \hat{\phi}_{1z} \\ \hat{d}_{2y} \\ \hat{\phi}_{2x} \\ \hat{\phi}_{2z} \end{Bmatrix}$$

The ***transformation matrix*** relating local to global degrees of freedom for a grid is:

$$T_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & S & 0 & 0 & 0 \\ 0 & -S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix}$$



where θ is now positive taken counterclockwise from x to $\hat{x}$ in the x-z plane: therefore;

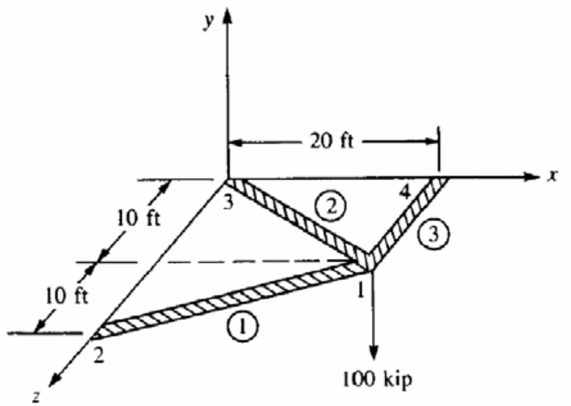
$$C = \cos\theta = \frac{x_j - x_i}{L} \qquad S = \sin\theta = \frac{z_j - z_i}{L}$$

The global stiffness matrix for a grid element arbitrary oriented in the x-z plane is given by:

$$k_G = T_G^T \hat{k}_G T_G$$

Grid Example

Consider the frame shown in the figure below.



The frame is fixed at nodes 2, 3, and 4, and is subjected to a load of 100 *kips* applied at node 1. Assume I = 400 in.^{4}, J = 110 in.^{4}, G = 12 x 10 ^{3} *ksi*, and E = 30 x 10 ^{3} *ksi* for all elements.

To facilitate a timely solution, the boundary conditions at nodes 2, 3, and 4 are applied to the local stiffness matrices at the beginning of the solution.

$$d_{2y} = \phi_{2x} = \phi_{2z} = 0$$
$$d_{3y} = \phi_{3x} = \phi_{3z} = 0$$
$$d_{4y} = \phi_{4x} = \phi_{4z} = 0$$

Beam Element 1:

$$C = \cos\theta = \frac{x_2 - x_1}{L^{(1)}} = \frac{0-20}{22.36} = -0.894 \qquad S = \sin\theta = \frac{z_2 - z_1}{L^{(1)}} = \frac{20-10}{22.36} = 0.447$$

where

$$\frac{12EI}{L^3} = \frac{12(30\times10^3)(400)}{(22.36\times12)^3} = 7.45\ {}^{k}\!/_{in} \qquad \frac{6EI}{L^2} = \frac{6(30\times10^3)(400)}{(22.36\times12)^2} = 1{,}000\ k$$

$$\frac{4EI}{L}=\frac{4(30\times10^3)(400)}{(22.36\times12)}=179{,}000\ k\cdot in \qquad \frac{GJ}{L}=\frac{(12\times10^3)(110)}{(22.36\times12)}=4{,}920\ k\cdot in$$

The global stiffness matrix for element 1, considering only the parts associated with node 1, and the following relationship:

$$k_G = T_G^{\,T}\hat{k}_G T_G$$

$$T_G=\begin{bmatrix}1&0&0\\0&-0.894&0.447\\0&-0.447&-0.894\end{bmatrix} \qquad T_G^{\,T}=\begin{bmatrix}1&0&0\\0&-0.894&-0.447\\0&0.447&-0.894\end{bmatrix}$$

Therefore, the global stiffness matrix is

$$\hat{k}^{(1)}=\begin{bmatrix}7.45&0&1{,}000\\0&4{,}920&0\\1{,}000&0&179{,}000\end{bmatrix}{}^{k}\!/_{in} \quad (d_{1y},\ \phi_{1x},\ \phi_{1z})$$

$$k^{(1)}=\begin{bmatrix}7.45&-447&-894\\-447&39{,}700&69{,}600\\-894&69{,}600&144{,}000\end{bmatrix}{}^{k}\!/_{in} \quad (d_{1y},\ \phi_{1x},\ \phi_{1z})$$

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Beam Element 2:

$$C=\cos\theta=\frac{x_3-x_1}{L^{(2)}}=\frac{0-20}{22.36}=-0.894 \qquad S=\sin\theta=\frac{z_3-z_1}{L^{(2)}}=\frac{0-10}{22.36}=-0.447$$

where

$$\frac{12EI}{L^3}=\frac{12(30\times10^3)(400)}{(22.36\times12)^3}=7.45\ {}^{k}\!/_{in} \qquad \frac{6EI}{L^2}=\frac{6(30\times10^3)(400)}{(22.36\times12)^2}=1{,}000\ k$$

$$\frac{4EI}{L}=\frac{4(30\times10^3)(400)}{(22.36\times12)}=179{,}000\ k\cdot in \qquad \frac{GJ}{L}=\frac{(12\times10^3)(110)}{(22.36\times12)}=4{,}920\ k\cdot in$$

The global stiffness matrix for element 2, considering only the parts associated with node 1, and the following relationship:

$$k_G = T_G^{\,T}\hat{k}_G T_G$$

$$k^{(2)}=\begin{bmatrix}1&0&0\\0&-0.894&0.447\\0&-0.447&-0.894\end{bmatrix}\begin{bmatrix}7.45&0&1{,}000\\0&4{,}920&0\\1{,}000&0&179{,}000\end{bmatrix}\begin{bmatrix}1&0&0\\0&-0.894&-0.447\\0&0.447&-0.894\end{bmatrix}=\begin{bmatrix}7.45&447&-894\\447&39{,}700&-69{,}600\\-894&-69{,}600&144{,}000\end{bmatrix}{}^{k}\!/_{in} \quad (d_{1y},\ \phi_{1x},\ \phi_{1z})$$

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Beam Element 3:

$$C=\cos\theta=\frac{x_4-x_1}{L^{(3)}}=\frac{20-20}{10}=0 \qquad S=\sin\theta=\frac{z_4-z_1}{L^{(3)}}=\frac{0-10}{10}=-1$$

where

$$\frac{12EI}{L^3}=\frac{12(30\times10^3)(400)}{(10\times12)^3}=83.3\ k/in \qquad \frac{6EI}{L^2}=\frac{6(30\times10^3)(400)}{(10\times12)^2}=5{,}000\ k$$

$$\frac{4EI}{L}=\frac{4(30\times10^3)(400)}{(10\times12)}=400{,}000\ k\cdot in \qquad \frac{GJ}{L}=\frac{(12\times10^3)(110)}{(10\times12)}=11{,}000\ k\cdot in$$

The global stiffness matrix for element 3, considering only the parts associated with node 1, and the following relationship:

$$k_G = T_G^{\,T}\hat{k}_G T_G$$

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$$k^{(3)}=\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}\begin{bmatrix}83.3&0&5{,}000\\0&11{,}000&0\\5{,}000&0&400{,}000\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$$

Therefore, the global stiffness matrix is

$$k_{(3)}=\begin{bmatrix}83.3&5{,}000&0\\5{,}000&400{,}000&0\\0&0&11{,}000\end{bmatrix} \quad (d_{1y},\ \phi_{1x},\ \phi_{1z})$$

Superimposing the three elemental stiffness matrices gives:

$$K=\begin{bmatrix}98.2&5{,}000&-1{,}790\\5{,}000&479{,}000&0\\-1{,}790&0&299{,}000\end{bmatrix} \quad (d_{1y},\ \phi_{1x},\ \phi_{1z})$$

The global equations are:

$$\begin{Bmatrix} F_{1y} = -100\ k \\ M_{1x} = 0 \\ M_{1z} = 0 \end{Bmatrix} = \begin{bmatrix} 98.2 & 5,000 & -1,790 \\ 5,000 & 479,000 & 0 \\ -1,790 & 0 & 299,000 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_{1x} \\ \phi_{1z} \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} d_{1y} \\ \phi_{1x} \\ \phi_{1z} \end{Bmatrix} = \begin{Bmatrix} -2.83\ in \\ 0.0295\ rad \\ -0.0169\ rad \end{Bmatrix}$$

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Element 1: The grid element force-displacement equations can be obtained using $\hat{f} = \hat{k}_G \overline{T}_G d$.

$$\overline{T}_G d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.894 & 0.447 & 0 & 0 & 0 \\ 0 & -0.447 & -0.894 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.894 & 0.447 \\ 0 & 0 & 0 & 0 & -0.447 & -0.894 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ 0.0295\ rad \\ -0.0169\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2.83\ in \\ -0.0339\ rad \\ 0.00192\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Therefore, the local force-displacement equations are:

$$\hat{f}_{(1)} = \hat{k}\overline{T}d = \begin{bmatrix} 7.45 & 0 & 1,000 & -7.45 & 0 & 1,000 \\ 0 & 4,920 & 0 & 0 & -4,920 & 0 \\ 1,000 & 0 & 179,000 & -1,000 & 0 & 89,500 \\ -7.45 & 0 & -1,000 & 7.45 & 0 & -1,000 \\ 0 & -4,920 & 0 & 0 & 4,920 & 0 \\ 1,000 & 0 & 89,500 & -1,000 & 0 & 179,000 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ -0.0339\ rad \\ 0.00192\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

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Element 2: The grid element force-displacement equations can be obtained using $\hat{f} = \hat{k}_G \overline{T}_G d$.

$$\overline{T}_G d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.894 & -0.447 & 0 & 0 & 0 \\ 0 & 0.447 & -0.894 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.894 & -0.447 \\ 0 & 0 & 0 & 0 & 0.447 & -0.894 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ 0.0295\ rad \\ -0.0169\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2.83\ in \\ -0.0188\ rad \\ 0.0283\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Therefore, the local force-displacement equations are:

$$\hat{f}_{(2)} = \hat{k}\overline{T}d = \begin{bmatrix} 7.45 & 0 & 1,000 & -7.45 & 0 & 1,000 \\ 0 & 4,920 & 0 & 0 & -4,920 & 0 \\ 1,000 & 0 & 179,000 & -1,000 & 0 & 89,500 \\ -7.45 & 0 & -1,000 & 7.45 & 0 & -1,000 \\ 0 & -4,920 & 0 & 0 & 4,920 & 0 \\ 1,000 & 0 & 89,500 & -1,000 & 0 & 179,000 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ -0.0188\ rad \\ 0.0283\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

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Element 3: The grid element force-displacement equations can be obtained using $\hat{f} = \hat{k}_G \overline{T}_G d$.

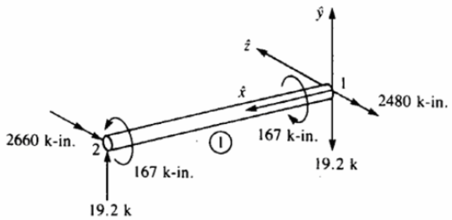
$$\overline{T}_G d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ 0.0295\ rad \\ -0.0169\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2.83\ in \\ 0.0169\ rad \\ 0.0295\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Therefore, the local force-displacement equations are:

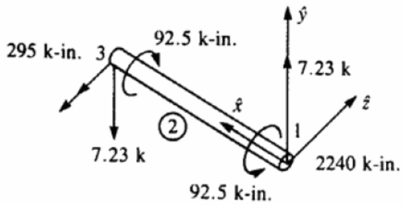
$$\hat{f}_{(3)} = \hat{k}\overline{T}d = \begin{bmatrix} 83.3 & 0 & 5,000 & -83.3 & 0 & 5,000 \\ 0 & 11,000 & 0 & 0 & -11,000 & 0 \\ 5,000 & 0 & 400,000 & -5,000 & 0 & 200,000 \\ -83.3 & 0 & -5,000 & 83.3 & 0 & -5,000 \\ 0 & -11,000 & 0 & 0 & 11,000 & 0 \\ 5,000 & 0 & 200,000 & -5,000 & 0 & 400,000 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ 0.0169\ rad \\ 0.0295\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solving for the forces and moments gives:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_{1x} \\ \hat{m}_{1z} \\ \hat{f}_{2y} \\ \hat{m}_{2x} \\ \hat{m}_{2z} \end{Bmatrix} = \begin{Bmatrix} -19.2\ k \\ -167\ k \cdot in \\ -2{,}480\ k \cdot in \\ 19.2\ k \\ 167\ k \cdot in \\ -2{,}260\ k \cdot in \end{Bmatrix}$$



$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_{1x} \\ \hat{m}_{1z} \\ \hat{f}_{3y} \\ \hat{m}_{3x} \\ \hat{m}_{3z} \end{Bmatrix} = \begin{Bmatrix} 7.23\ k \\ -92.5\ k \cdot in \\ -2{,}240\ k \cdot in \\ -7.23\ k \\ 92.5\ k \cdot in \\ -295\ k \cdot in \end{Bmatrix}$$



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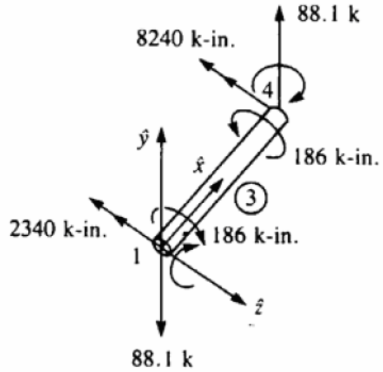
Solving for the forces and moments gives:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_{1x} \\ \hat{m}_{1z} \\ \hat{f}_{4y} \\ \hat{m}_{4x} \\ \hat{m}_{4z} \end{Bmatrix} = \begin{Bmatrix} -88.1\ k \\ 186\ k \cdot in \\ -2{,}340\ k \cdot in \\ 88.1\ k \\ -186\ k \cdot in \\ -8{,}240\ k \cdot in \end{Bmatrix}$$



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To check the equilibrium of node 1 the local forces and moments for each element need to be transformed to global coordinates. Recall, that:

$$\hat{f} = Tf \quad \Rightarrow \quad f = T^T\hat{f} \qquad T^T = T^{-1}$$

Since we are only checking the forces and moments at node 1, we need only the upper-left-hand portion of the transformation matrix $\boldsymbol{T_G}$.

Therefore; for **Element 1:**

$$\begin{Bmatrix} f_{1y} \\ m_{1x} \\ m_{1z} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & -0.447 \\ 0 & 0.447 & -0.894 \end{bmatrix} \begin{Bmatrix} -19.2\ k \\ -167\ k \cdot in \\ -2{,}480\ k \cdot in \end{Bmatrix} = \begin{Bmatrix} -19.2\ k \\ 1{,}260\ k \cdot in \\ 2{,}150\ k \cdot in \end{Bmatrix}$$

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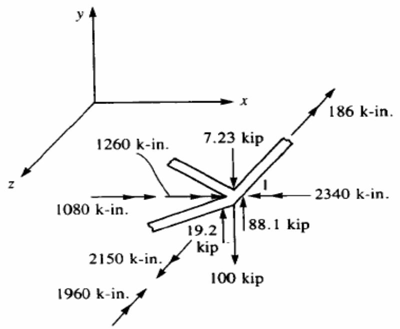
Therefore; for **Element 2:**

$$\begin{Bmatrix} f_{1y} \\ m_{1x} \\ m_{1z} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & 0.447 \\ 0 & -0.447 & -0.894 \end{bmatrix} \begin{Bmatrix} -7.23\ k \\ -92.5\ k \cdot in \\ -2{,}240\ k \cdot in \end{Bmatrix} = \begin{Bmatrix} 7.23\ k \\ 1{,}080\ k \cdot in \\ -1{,}960\ k \cdot in \end{Bmatrix}$$

Therefore; for **Element 3:**

$$\begin{Bmatrix} f_{1y} \\ m_{1x} \\ m_{1z} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} -88.1\ k \\ -2{,}340\ k \cdot in \\ -186\ k \cdot in \end{Bmatrix} = \begin{Bmatrix} -88.1\ k \\ -2{,}340\ k \cdot in \\ -186\ k \cdot in \end{Bmatrix}$$

The forces and moments that are applied to node 1 by each element are equal in magnitude and opposite direction. Therefore the sum of the forces and moments acting on node 1 are:



$$\sum F_{1y} = -100 - 7.23 + 19.2 + 88.1 = 0.07\ k$$

$$\sum M_{1x} = -1{,}260 - 1{,}080 + 2{,}340 = 0.0\ k \cdot in$$

$$\sum M_{1z} = -2{,}150 + 1{,}060 + 186 = -4.0\ k \cdot in$$

The forces and moments accurately satisfy equilibrium considering the amount of truncation error inherent in results of the calculations presented in this example.

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Stiffness matrix of a beam element oriented in space

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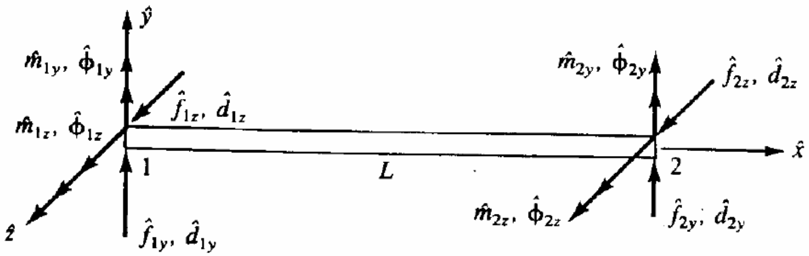
Beam Element Arbitrarily Oriented in Space

In this section,we will develop a beam element that is arbitrarily oriented in three-dimensions. This element can be used to analyze three-dimensional frames. Let consider bending about axes, as shown below.



The $\hat{y}$ axis is the principle axis for which the moment of inertia is minimum, I_y. The right-hand rule is used to establish the $\hat{z}$ axis and the maximum moment of inertia, I_z.

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Bending in the $\hat{x} - \hat{z}$ plane: The bending in the $\hat{x} - \hat{z}$ plane is defined by $\hat{m}_y$. The stiffness matrix for bending the in the *x-z* plane is:

$$\hat{k}_y = \frac{EI_y}{L^4}\begin{bmatrix} 12L & 6L^2 & -12L & 6L^2 \\ 6L^2 & 4L^3 & -6L^2 & 2L^3 \\ -12L & -6L^2 & 12L & -6L^2 \\ 6L^2 & 2L^3 & -6L^2 & 4L^3 \end{bmatrix}$$

where I_y is the moment of inertia about the $\hat{y}$ axis (the weak axis).

Bending in the $\hat{x} - \hat{y}$ plane: The bending in the $\hat{x} - \hat{y}$ plane is defined by $\hat{m}_z$. The stiffness matrix for bending the in the $\hat{x} - \hat{y}$ plane is:

$$\hat{k}_z = \frac{EI_z}{L^4}\begin{bmatrix} 12L & 6L^2 & -12L & 6L^2 \\ 6L^2 & 4L^3 & -6L^2 & 2L^3 \\ -12L & -6L^2 & 12L & -6L^2 \\ 6L^2 & 2L^3 & -6L^2 & 4L^3 \end{bmatrix}$$

where I_z is the moment of inertia about the $\hat{z}$ axis (the strong axis).

Direct superposition of the bending stiffness matrices with the effects of axial forces and torsional rotation give:

$$\hat{k} = \begin{array}{c} \\ \end{array}\left[\begin{array}{cccccc|cccccc}
\hat{d}_{1x} & \hat{d}_{1y} & \hat{d}_{1z} & \hat{\phi}_{1x} & \hat{\phi}_{1y} & \hat{\phi}_{1z} & \hat{d}_{2x} & \hat{d}_{2y} & \hat{d}_{2z} & \hat{\phi}_{2x} & \hat{\phi}_{2y} & \hat{\phi}_{2z} \\
\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
\hline
-\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{array}\right]$$

The global stiffness matrix may be obtained using:

$$k = T^T \hat{k} T$$

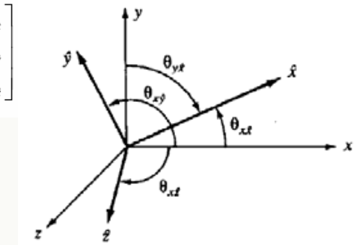
where

$$T = \begin{bmatrix} \lambda_{3\times3} & & & \\ & \lambda_{3\times3} & & \\ & & \lambda_{3\times3} & \\ & & & \lambda_{3\times3} \end{bmatrix}$$

where

where the direction cosines, C_{ij}, are defined as shown below

$$\lambda_{3\times3} = \begin{bmatrix} C_{x\hat{x}} & C_{y\hat{x}} & C_{z\hat{x}} \\ C_{x\hat{y}} & C_{y\hat{y}} & C_{z\hat{y}} \\ C_{x\hat{z}} & C_{y\hat{z}} & C_{z\hat{z}} \end{bmatrix}$$



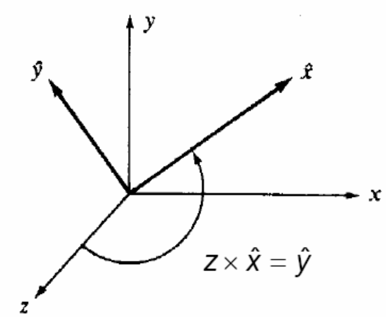
The direction cosines of the $\hat{x}$ axis are:

$$\hat{x} = \cos\theta_{x\hat{x}}\,\bar{i} + \cos\theta_{y\hat{x}}\,\bar{j} + \cos\theta_{z\hat{x}}\,\bar{k}$$

where

$$\cos\theta_{x\hat{x}} = \frac{x_2 - x_1}{L} = l \qquad \cos\theta_{y\hat{x}} = \frac{y_2 - y_1}{L} = m \qquad \cos\theta_{z\hat{x}} = \frac{z_2 - z_1}{L} = n$$

The $\hat{y}$ axis is selected to be perpendicular to the $\hat{x}$ and the z axes is such a way that the cross product of global z with $\hat{x}$ results in the $\hat{y}$ axis as shown in the figure below.



$$z \times \hat{x} = \hat{y} = \frac{1}{D}\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & 1 \\ l & m & n \end{vmatrix} = -\frac{m}{D}\bar{i} + \frac{l}{D}\bar{j}$$

where

$$D = \sqrt{l^2 + m^2}$$

The $\hat{z}$ axis is determined by the condition that $\hat{z} = \hat{x} \times \hat{y}$

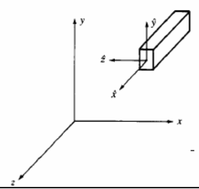
$$\hat{z} = \hat{x} \times \hat{y} = \frac{1}{D}\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ l & m & n \\ -m & l & 0 \end{vmatrix} = -\frac{ln}{D}\bar{i} - \frac{mn}{D}\bar{j} + D\bar{k}$$

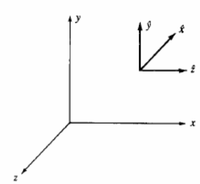
Therefore, the transformation matrix becomes:

$$\lambda_{3\times3} = \begin{bmatrix} C_{x\hat{x}} & C_{y\hat{x}} & C_{z\hat{x}} \\ C_{x\hat{y}} & C_{y\hat{y}} & C_{z\hat{y}} \\ C_{x\hat{z}} & C_{y\hat{z}} & C_{z\hat{z}} \end{bmatrix} \qquad \lambda_{3\times3} = \begin{bmatrix} l & m & n \\ -\frac{m}{D} & \frac{l}{D} & 0 \\ -\frac{ln}{D} & \frac{mn}{D} & D \end{bmatrix}$$

There are two exceptions that arise when using the above expressions for mapping the local coordinates to the global system: (1) when the positive $\hat{x}$ coincides with z; and (2) when the positive $\hat{x}$ is in the opposite direction as z. For the first case, it is assumed that $\hat{y}$ is y. In case two, it is assumed that $\hat{y}$ is y.

$$\lambda = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \lambda = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$





If the effects of axial force, both shear forces, twisting moment, and both bending moments are considered, the stiffness matrix for a frame element is:

$$
\begin{bmatrix}
\frac{EA}{l} & & & & & & & & & & & \\
0 & \frac{12EI_z}{l^3(1+\Phi_y)} & & & & & & & & & & \\
0 & 0 & \frac{12EI_y}{l^3(1+\Phi_z)} & & & & & & & & & \\
0 & 0 & 0 & \frac{GJ}{l} & & & & \text{Symmetric} & & & & \\
0 & 0 & \frac{-6EI_y}{l^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_y}{l(1+\Phi_z)} & & & & & & & \\
0 & \frac{6EI_z}{l^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(4+\Phi_y)EI_z}{l(1+\Phi_y)} & & & & & & \\
\frac{-EA}{l} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{l} & & & & & \\
0 & \frac{-12EI_z}{l^3(1+\Phi_y)} & 0 & 0 & 0 & \frac{-6EI_z}{l^2(1+\Phi_y)} & 0 & \frac{12EI_z}{l^3(1+\Phi_y)} & & & & \\
0 & 0 & \frac{-12EI_y}{l^3(1+\Phi_z)} & 0 & \frac{6EI_y}{l^2(1+\Phi_z)} & 0 & 0 & 0 & \frac{12EI_y}{l^3(1+\Phi_z)} & & & \\
0 & 0 & 0 & \frac{-GJ}{l} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{l} & & \\
0 & 0 & \frac{-6EI_y}{l^2(1+\Phi_z)} & 0 & \frac{(2-\Phi_z)EI_y}{l(1+\Phi_z)} & 0 & 0 & 0 & \frac{6EI_y}{l^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_y}{l(1+\Phi_z)} & \\
0 & \frac{6EI_z}{l^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(2-\Phi_y)EI_z}{l(1+\Phi_y)} & 0 & \frac{-6EI_z}{l^2(1+\Phi_y)} & 0 & 0 & 0 & \frac{(4+\Phi_y)EI_z}{l(1+\Phi_y)}
\end{bmatrix}
$$

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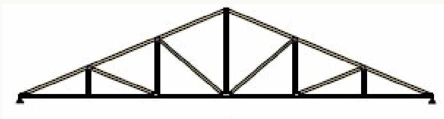
In this case the symbol ϕ are:

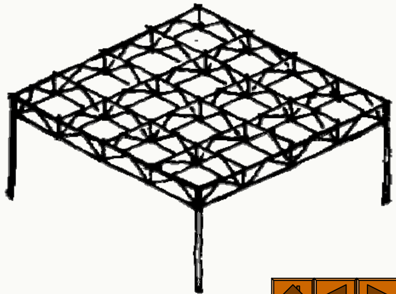
$$\phi_y = \frac{12EI_y}{GA_sL^2} \qquad \phi_z = \frac{12EI_z}{GA_sL^2}$$

where A_s is the effective beam cross-section in shear. Recall the shear modulus of elasticity or the modulus of rigidity, G, is related to the modulus of elasticity and the Poisson's ratio, ν as:

$$G = \frac{E}{2(1+\nu)}$$

- So far, we considered only line elements.
- Line elements are connected at common nodes, forming trusses, frames, and grids.





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- Line elements have geometric properties (A, I associate with cross sections).
- Only one local coordinate along the length of the element is required to describe a position along the line element.
- Nodal compatibility is forced during the formulation of the nodal equilibrium equations for a line element.

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2D Finite Elements

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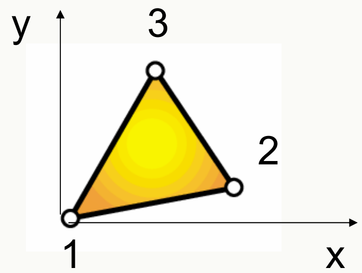
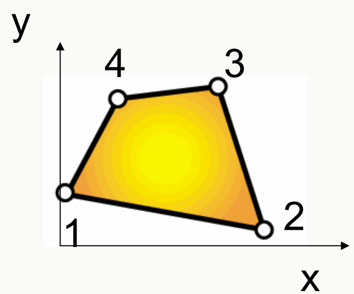
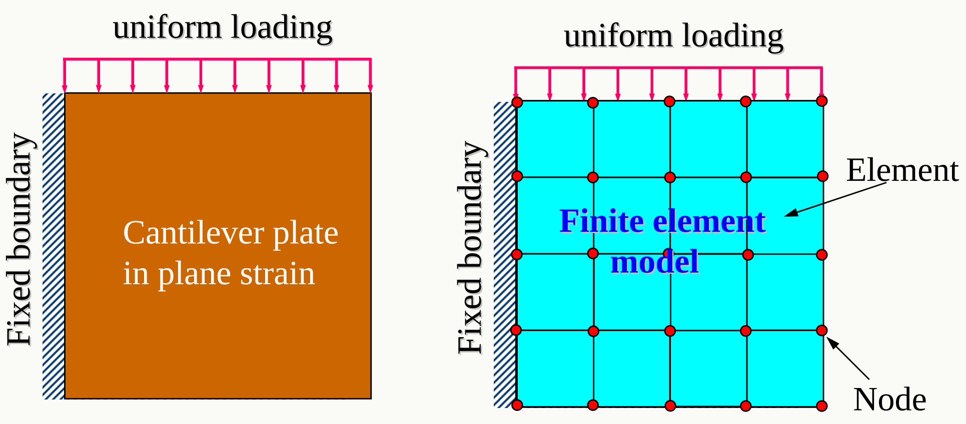
- 2D planar elements are thin-plate elements.
- 2 coordinates define a position on the 2D element surface.







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- 2D elements are connected at common nodes and/or along common edges to form continuous structures.
- Nodal compatibility is enforced during the formulation of the nodal equilibrium equations.
- If proper displacement functions are chosen, compatibility along common edges is obtained.

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The 2D elements are extremely important for:

- **Plane stress analysis**: problems such as plates with holes or other changes in geometry that are loaded in plane resulting in local stress concentrations.
- **Plane strain analysis**: problems such as long underground box culvert subjected to a uniform loading acting constantly over its length.

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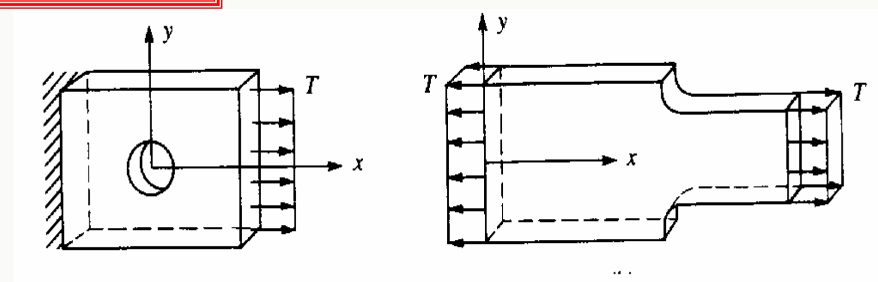
Plane Stress

Plane stress *is defined to be a state of stress in which the normal stress and the shear stresses directed perpendicular to the plane are assumed to be zero.* That is, the normal stress σ_z and the shear stresses τ_{xz} and τ_{yz} are assumed to be zero. Generally, members that are thin (those with a small z dimension compared to the in-plane x and y dimensions) and whose loads act only in the x-y plane can be considered to be under plane stress.

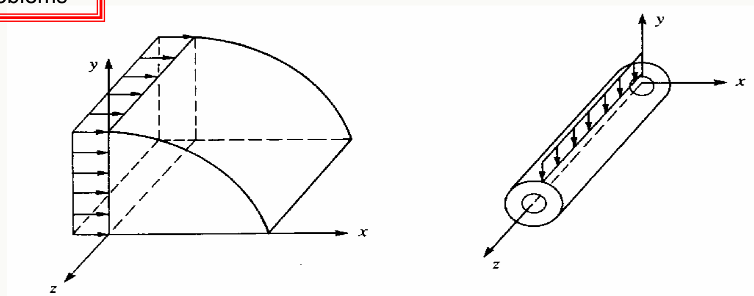
Plane Stress Problems



Plane Strain

Plane strain *is defined to be a state of strain in which the strain normal to the x-y plane* ε_z *and the shear strains* γ_{xz} *and* γ_{yz} *are assumed to be zero*. The assumptions of plane strain are realistic for long bodies (say, in the *z* direction) with constant cross-sectional area subjected to loads that act only in the *x* and/or *y* directions and do not vary in the *z* direction.

Plane Strain problems



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• Direct Stiffness – 2D FEMs

Summary:

- **The review of the principle of minimum potential energy.**
- **The development of the stiffness matrix of a basic 2D or plane finite element called Constant-Strain Triangular (CST) elements.**
- **Example Problems.**

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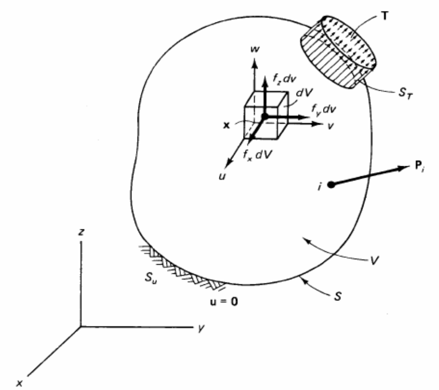
Potential Energy and Equilibrium

In mechanics of solids, our problem is to determine the displacement ***u*** of the body, satisfying the equilibrium equations.



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The principle of minimum potential energy

Total Potential Energy

The total potential energy is defined as the sum of the internal strain energy ***U*** and the potential energy of the external forces Ω:

$$\pi_p = U + \Omega$$

Strain energy is the capacity of the internal forces (or stresses) to do work through deformations (strains) in the structure; Ω is the capacity of forces such as body forces, surface traction forces, and applied nodal forces to do work through the deformation of the structure.

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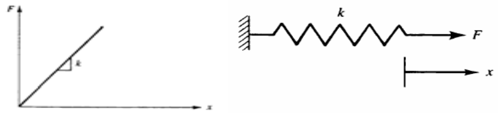
Recall the force-displacement relationship for a linear spring:

$$F = kx$$

The differential internal work (or strain energy) ***dU*** in the spring is the internal force multiplied by the change in displacement which the force moves through:

$$dU = Fdx = (kx)dx$$



The total strain energy is:

$$U = \int dU = \int_0^x (kx)\,dx = \frac{1}{2}kx^2$$

The strain energy is the area under the force-displacement curve. The potential energy of the external forces is the work done by the external forces: $\Omega = -Fx$

Therefore, the total potential energy is:

$$\pi_p = \frac{1}{2}kx^2 - Fx$$

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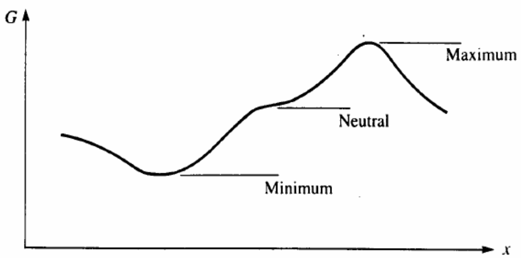
The concept of a ***stationary value*** of a function ***G*** is shown below:



The function ***G*** is expressed in terms of ***x***. To find a value of ***x*** yielding a stationary value of ***G***(*x*), we use differential calculus to differentiate ***G*** with respect to *x* and set the expression equal to zero.

$$\frac{dG}{dx} = 0$$

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We can replace ***G*** with the total potential energy π_p and the coordinate *x* with a discrete value d_i. To minimize π_p we first take the ***variation*** of π_p (we will not cover the details of variational calculus):

$$\delta\pi_p = \frac{\partial \pi_p}{\partial d_1}\delta d_1 + \frac{\partial \pi_p}{\partial d_2}\delta d_2 + \ldots + \frac{\partial \pi_p}{\partial d_n}\delta d_n$$

The principle states that equilibrium exist when the $\boldsymbol{d}_i$ define a structure state such that $\delta\pi_p = 0$ for arbitrary admissible variations $\delta\boldsymbol{d}_i$ from the equilibrium state. An ***admissible variation*** is one in which the displacement field still satisfies the boundary conditions and interelement continuity.

To satisfy $\delta\pi_p = 0$, all coefficients associated with $\delta\boldsymbol{d}_i$ must be zero independently, therefore:

$$\frac{\partial \pi_p}{\partial d_i} = 0 \qquad i = 1, 2, \cdots, n \qquad or \qquad \frac{\partial \pi_p}{\partial \{d\}} = 0$$

Let's assume F=1000 lb, k=500 lb/in. The total potential energy is defined as $\pi_p = U + \Omega$ $U = \frac{1}{2}kx^2$ $\Omega = -Fx$

The variation of π_p with respect to x is:

$$\delta\pi_p = \frac{\partial \pi_p}{\partial x}\delta x = 0$$

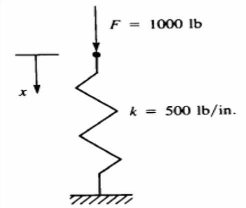
Since δx is arbitrary and might not be zero, then:

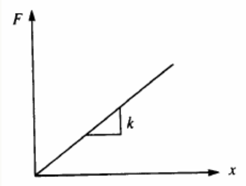
$$\frac{\partial \pi_p}{\partial x} = 0$$

Using our express for π_p, we get:

$$\pi_p = \frac{1}{2}kx^2 - Fx = \frac{1}{2}500(lb/in)\,x^2 - (1000lb)\,x$$

$$\frac{\partial \pi_p}{\partial x} = 0 = 500x - 1000 \qquad x = 2.0in.$$





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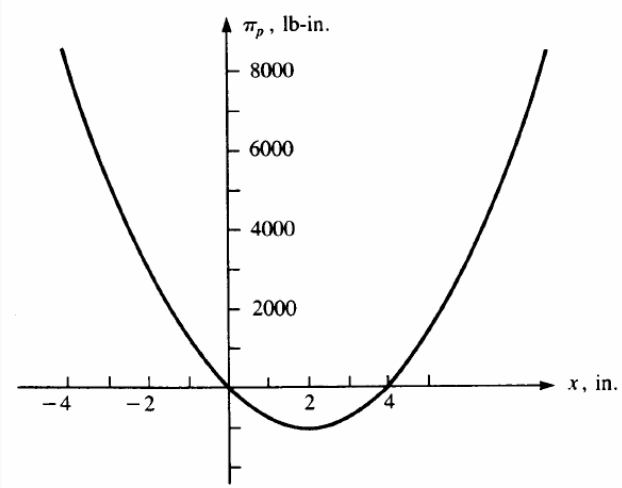
If we had plotted the total potential energy function for various values of deformation, we would get:



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Now let's derive the spring element equations and stiffness matrix using the principal of minimum potential energy. Consider the linear spring subjected to nodal forces shown below:

The total potential energy π_p is:

$$\pi_p = \frac{1}{2}k(\hat{d}_{2x} - \hat{d}_{1x})^2 - \hat{f}_{1x}\hat{d}_{1x} - \hat{f}_{2x}\hat{d}_{2x}$$

Expanding the above express gives:

$$\pi_p = \frac{1}{2}k(\hat{d}_{2x}^2 - 2\hat{d}_{1x}\hat{d}_{2x} - \hat{d}_{1x}^2) - \hat{f}_{1x}\hat{d}_{1x} - \hat{f}_{2x}\hat{d}_{2x}$$

Minimizing the total potential energy π_p:

$$\frac{\partial \pi_p}{\partial d_i} = 0 \qquad i = 1 \text{ to } 2$$

$$\frac{\partial \pi_p}{\partial d_1} = 0 = \frac{1}{2}k(-2\hat{d}_{2x} + 2\hat{d}_{1x}) - \hat{f}_{1x}$$

$$\frac{\partial \pi_p}{\partial d_2} = 0 = \frac{1}{2}k(2\hat{d}_{2x} - 2\hat{d}_{1x}) - \hat{f}_{2x}$$

In matrix form the above equations are:

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

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The development of the stiffness matrix of a basic 2D or plane finite element

To fully understand the development and applicability of the stiffness matrix for the plane stress/plane strain triangular element, the concept of 2D state of stress and strain and the stress/strain relationships for plane stress and plane strain are necessary.

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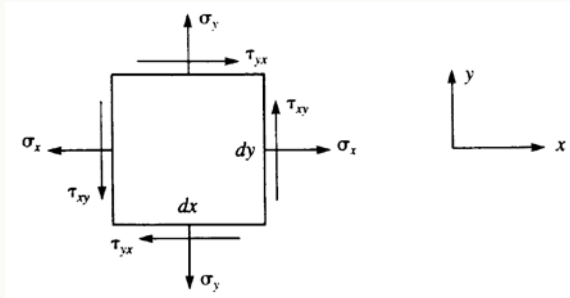
Two-Dimensional State of Stress and Strain



Since τ_{xy} equals τ_{yx}, three independent stress exist:

$$\{\sigma\}^T = \begin{bmatrix} \sigma_x & \sigma_y & \tau_{xy} \end{bmatrix}$$

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Recall, the relationships for ***principal stresses*** in two-dimensions are:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} = \sigma_{\max}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} = \sigma_{\min}$$

Also, θ_p is the ***principal angle*** which defines the normal whose direction is perpendicular to the plane on which the maximum or minimum principle stress acts.

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

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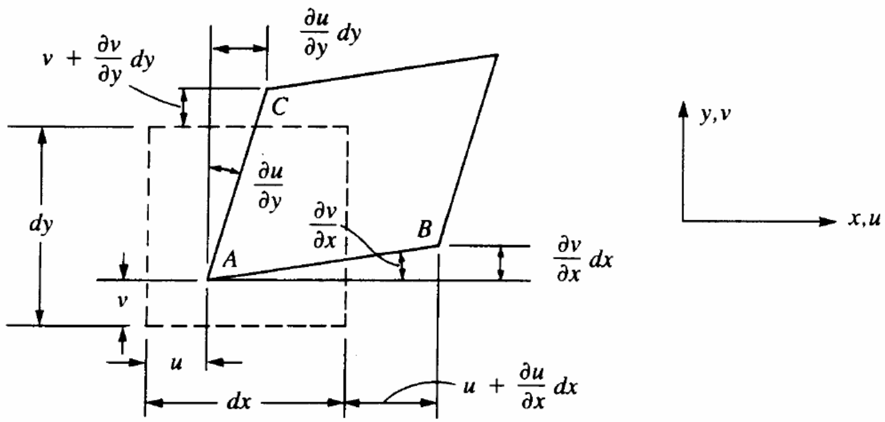
The general two-dimensional state of strain at a point is show below.



The general definitions of normal and shear strains are:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

The strain may be written in matrix form as:

$$\{\varepsilon\}^T = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{bmatrix}$$

Plane Stress

For **plane stress,** the stresses σ_z, τ_{xz}, and τ_{yz} are assumed to be zero.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

where

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

← stress-strain matrix (constitutive matrix)

Plane Strain

For **plane strain**, the strains ε_z, γ_{xz}, and γ_{yz} are assumed to be zero.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

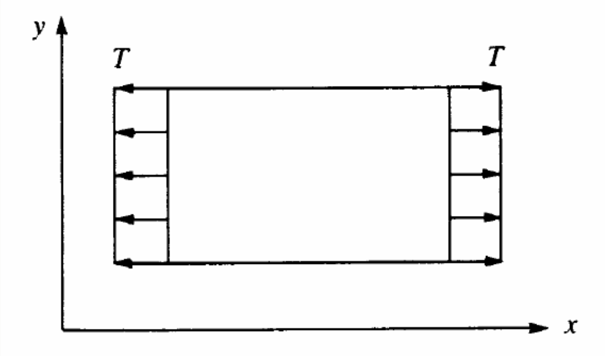
where

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

← stress-strain matrix (constitutive matrix)

Steps in the formulation of element stiffness equations

Consider the problem of a thin plate subjected to a tensile load as shown below



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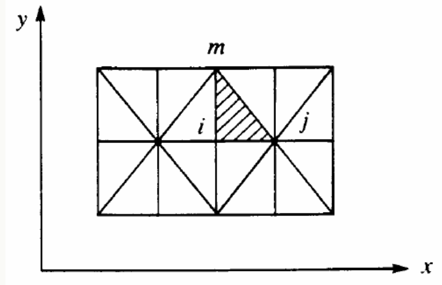
Step 1 : Discretize and Select element types

Discretize the thin plate into a set of triangular elements. Each element is defined by nodes i, j, and m. Each node has 2 DOFs (displacements in x-,y-directions)



We use triangular elements because

1. Boundaries of irregularly shape bodies can be closely approximated.
2. The expressions related to the triangular element are simple.

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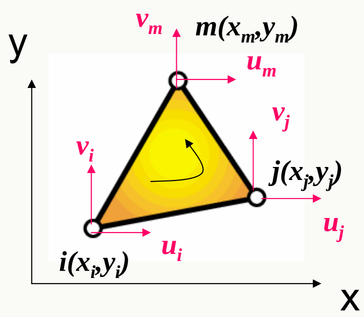
let u_i and v_i represent the node i displacement components in the x and y directions, respectively.

The nodal displacements for an element with nodes i, j, and m are:



$$\{d\} = \begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix} \quad \text{where} \quad \{d_i\} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

Therefore:

$$\{d\} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

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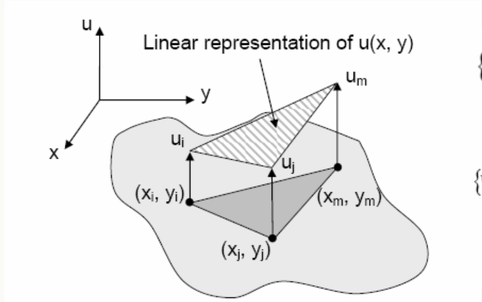
Step 2 : Select Displacement Functions

A linear displacement function is selected for each triangular element, defined as



$$\{\Psi_i\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix}$$

$$\{\Psi_i\} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

A linear displacement function ensures that the displacements along each edge of the element and the nodes shared by adjacent elements are equal.

To obtain the values for the ***a***'s substitute the coordinated of the nodal points into the above equations:

$$u_i = a_1 + a_2 x_i + a_3 y_i \qquad v_i = a_4 + a_5 x_i + a_6 y_i$$
$$u_j = a_1 + a_2 x_j + a_3 y_j \qquad v_j = a_4 + a_5 x_j + a_6 y_j$$
$$u_m = a_1 + a_2 x_m + a_3 y_m \qquad v_m = a_4 + a_5 x_m + a_6 y_m$$

Solving for the ***a***'s and writing the results in matrix forms gives:

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \quad \Rightarrow \quad \{a\} = [x]^{-1}\{u\}$$

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The inverse of the **[*x*]** matrix is:

$$[x]^{-1} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}$$

where

$$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

is the determinant of **[*x*]**.

$$2A = x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j)$$

where ***A*** is the area of the triangle and

$$\alpha_i = x_j y_m - y_j x_m \qquad \beta_i = y_j - y_m \qquad \gamma_i = x_m - x_j$$
$$\alpha_j = x_i y_m - y_i x_m \qquad \beta_j = y_m - y_i \qquad \gamma_j = x_i - x_m$$
$$\alpha_m = x_i y_j - y_i x_j \qquad \beta_m = y_i - y_j \qquad \gamma_m = x_j - x_i$$

The values of ***a*** may be written matrix form as:

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

and

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} v_i \\ v_j \\ v_m \end{Bmatrix}$$

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We will now derive the displacement function in terms of the coordinates *x* and *y*.

$$\{u\} = \begin{bmatrix} 1 & x & y \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Substituting the values for ***a*** into the above equation gives:

$$\{u\} = \frac{1}{2A}\begin{bmatrix} 1 & x & y \end{bmatrix}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

Expanding the above equations

$$\{u\} = \frac{1}{2A}\begin{bmatrix} 1 & x & y \end{bmatrix}\begin{bmatrix} \alpha_i u_i + \alpha_j u_j + \alpha_m u_m \\ \beta_i u_i + \beta_j u_j + \beta_m u_m \\ \gamma_i u_i + \gamma_j u_j + \gamma_m u_m \end{bmatrix}$$

Multiplying the matrices in the above equations gives

$$u(x,y) = \frac{1}{2A}\left\{(\alpha_i + \beta_i x + \gamma_i y)u_i + (\alpha_j + \beta_j x + \gamma_j y)u_j + (\alpha_m + \beta_m x + \gamma_m y)u_m\right\}$$

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A similar expression can be obtained for the *y* displacement

$$v(x,y) = \frac{1}{2A}\left\{(\alpha_i + \beta_i x + \gamma_i y)v_i + (\alpha_j + \beta_j x + \gamma_j y)v_j + (\alpha_m + \beta_m x + \gamma_m y)v_m\right\}$$

The displacements can be written in a more convenience form as:

$$u(x,y) = N_i u_i + N_j u_j + N_m u_m \qquad v(x,y) = N_i v_i + N_j v_j + N_m v_m$$

where

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y) \qquad N_j = \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y) \qquad N_m = \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)$$

The elemental displacements can be summarized as:

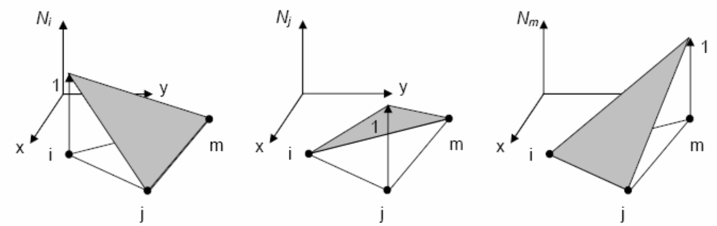
$$\{\Psi_i\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}$$

In another form the above equations are:

$$\{\Psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \qquad \{\psi\} = [N]\{d\}$$

where $[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$

The linear triangular shape functions are illustrated below



Step 3 : Define the Strain-Displacement and Stress-Strain Relationships

Elemental Strains: The strains over a 2D element are:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

Substituting our approximation for the displacement gives:

$$\frac{\partial u}{\partial x} = u_{,x} = \frac{\partial}{\partial x}\left(N_i u_i + N_j u_j + N_m u_m\right)$$

$$u_{,x} = N_{i,x} u_i + N_{j,x} u_j + N_{m,x} u_m$$

where the comma indicates differentiation with respect to that variable. The derivatives of the interpolation functions are:

$$N_{i,x} = \frac{1}{2A}\frac{\partial}{\partial x}\left(\alpha_i + \beta_i x + \gamma_i y\right) = \frac{\beta_i}{2A} \qquad N_{j,x} = \frac{\beta_j}{2A} \qquad N_{m,x} = \frac{\beta_m}{2A}$$

Therefore:

$$\frac{\partial u}{\partial x} = \frac{1}{2A}\left(\beta_i u_i + \beta_j u_j + \beta_m u_m\right)$$

In a similar manner, the remaining strain terms are approximated as:

$$\frac{\partial v}{\partial y} = \frac{1}{2A}\left(\gamma_i v_i + \gamma_j v_j + \gamma_m v_m\right)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}\left(\beta_i u_i + \gamma_i v_i + \beta_j u_j + \gamma_j v_j + \beta_m u_m + \gamma_m v_m\right)$$

We can write the strains in matrix form as:

$$\{\varepsilon\} == \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{2A}\left[\begin{array}{cc|cc|cc} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{array}\right] \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

or

$$\{\varepsilon\} = \begin{bmatrix} B_i & B_j & B_m \end{bmatrix} \begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix}$$

where

$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \qquad [B_j] = \frac{1}{2A}\begin{bmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix} \qquad [B_m] = \frac{1}{2A}\begin{bmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{bmatrix}$$

These equations can be written in matrix form as:

$$\{\varepsilon\} = [B]\{d\}$$

Stress-Strain Relationship: The in-plane stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

where $[D]$ for plane stress is:

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

and $[D]$ for plane strain is:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

In-plane stress can be related to displacements by:

$$\{\sigma\} = [D][B]\{d\}$$

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Step 4: Derive the Element Stiffness Matrix and Equations using the Total Potential Energy Approach

The total potential energy is defined as the sum of the internal strain energy U and the potential energy of the external forces Ω :

$$\pi_p = U + \Omega_b + \Omega_p + \Omega_s$$

where the strain energy is:

$$U = \frac{1}{2}\int_V \{\varepsilon\}^T \{\sigma\} dV \rightarrow U = \frac{1}{2}\int_V \{\varepsilon\}^T [D]\{\varepsilon\} dV$$

The potential energy of the ***body force*** term is:

$$\Omega_b = -\int_V \{\Psi\}^T \{X\} dV$$

where $\{\Psi\}$ is the general displacement function, and $\{X\}$ is the body weight per unit volume.

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The potential energy of the ***concentrated forces*** is:

$$\Omega_p = -\{d\}^T \{P\}$$

where $\{P\}$ are the concentrated forces, and $\{d\}$ are the nodal displacements.

The potential energy of the ***distributed loads*** is:

$$\Omega_s = -\int_S \{\Psi\}^T \{T\} dS$$

where $\{\Psi\}$ is the general displacement function, and $\{T\}$ are the surface tractions.

Then the total potential energy expression becomes:

$$\pi_p = \frac{1}{2}\int_V \{d\}^T [B]^T [D][B]\{d\}\, dV - \int_V \{d\}^T [N]^T \{X\} dV - \{d\}^T \{P\} - \int_S \{d\}^T [N]^T \{T\} dS$$

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The nodal displacements $\{d\}$ are independent of the general x-y coordinates, therefore

$$\pi_p = \frac{1}{2}\{d\}^T \int_V [B]^T [D][B] dV \{d\} - \{d\}^T \int_V [N]^T \{X\} dV - \{d\}^T \{P\} - \{d\}^T \int_S [N]^T \{T\} dS$$

We can define the last three terms as:

$$\{f\} = \int_V [N]^T \{X\} dV + \{P\} + \int_S [N]^T \{T\} dS$$

Therefore:

$$\pi_p = \frac{1}{2}\{d\}^T \int_V [B]^T [D][B] dV \{d\} - \{d\}^T \{f\}$$

Minimization of π_p with respect to each nodal displacement requires that:

$$\frac{\partial \pi_p}{\partial \{d\}} = \int_V [B]^T [D][B] dV \{d\} - \{f\} = 0$$

The above relationship requires:

$$\int_V [B]^T [D][B] dV \{d\} = \{f\}$$

The stiffness matrix can be defined as:

$$[k] = \int_V [B]^T [D][B] dV$$

For an element of constant thickness, t, the above integral becomes:

$$[k] = t \int_A [B]^T [D][B] \, dx \, dy$$

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The integrand in the above equation is not a function of x or y (global coordinates); therefore, the integration reduces to:

$$[k] = t[B]^T [D][B] \int_A dx \, dy \qquad \text{or} \qquad [k] = tA \, [B]^T [D][B]$$

where A is the area of the triangular element. Expanding the stiffness relationship gives:

$$[k] = \begin{bmatrix} [k_{ii}] & [k_{ij}] & [k_{im}] \\ [k_{ji}] & [k_{jj}] & [k_{jm}] \\ [k_{mi}] & [k_{mj}] & [k_{mm}] \end{bmatrix}$$

where each $[k_{ii}]$ is a 2 x 2 matrix define as:

$$[k_{ii}] = [B_i]^T [D][B_i] \, tA \qquad [k_{ij}] = [B_i]^T [D][B_j] \, tA \qquad [k_{im}] = [B_i]^T [D][B_m] \, tA$$

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Recall:

$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \qquad [B_j] = \frac{1}{2A}\begin{bmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix} \qquad [B_m] = \frac{1}{2A}\begin{bmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{bmatrix}$$

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Step 5: Assemble the Element Equations to obtain the Global Equations and Introduce the Boundary Conditions

The global stiffness matrix can be found by the direct stiffness method.

$$[K] = \sum_{e=1}^{N} [k^{(e)}]$$

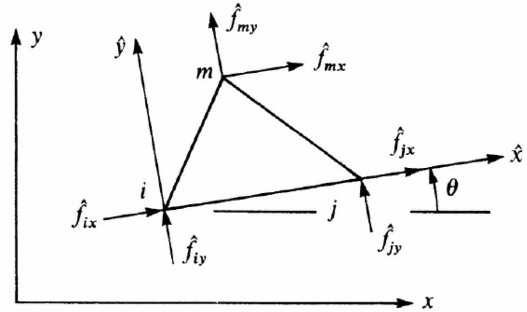
The global equivalent nodal load vector is obtained by lumping body forces and distributed loads at the appropriate nodes as well as including any concentrated loads.

$$\{F\} = \sum_{e=1}^{N} \{f^{(e)}\}$$

The resulting global equations are: $\{F\} = [K]\{d\}$

where $\{d\}$ is the total structural displacement vector.

In the above formulation of the element stiffness matrix, the matrix has been derived for a general orientation in global coordinates. Therefore, no transformation form local to global coordinates is necessary. However, for completeness, we will now describe the method to use if the local axes for the constant-strain triangular element are not parallel to the global axes for the whole structure.



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To relate the local to global displacements, force, and stiffness matrices we will use:

$$\hat{d} = Td \qquad \hat{f} = Tf \qquad k = T^T \hat{k} T$$

The transformation matrix T for the triangular element is:

$$T = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & C & S & 0 & 0 \\ 0 & 0 & -S & C & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix}$$

where $C = \cos\theta$ and $S = \sin\theta$, and θ is shown in the figure above.

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Step 6: Solve for Nodal Displacements
Step 7: Solve for Element Forces and Stresses

Having solved for the nodal displacements, we can obtain strains and stresses in x and y directions in the elements by using:

$$\{\varepsilon\} = [B]\{d\} \qquad \{\sigma\} = [D][B]\{d\}$$

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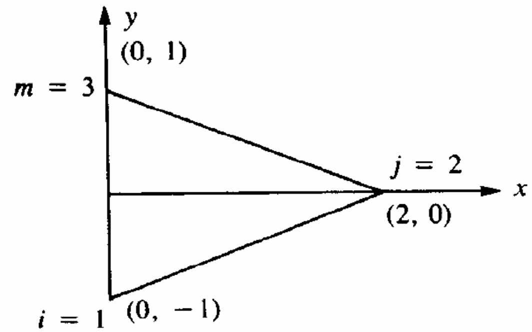
Consider the structure shown in the figure below.



Assume plane stress conditions. All coordinates are shown on the figure. Let E = 30 x 10^6 psi, ν = 0.25, and t = 1 in. Assume the element nodal displacements have been determined to be u_1 = 0.0, v_1 = 0.0025 in., u_2 = 0.0012 in., v_2 = 0.0, u_3 = 0.0, and v_3 = 0.0025 in. Determine the element stiffness matrix and the element stresses.

First, we calculate the element βs and γs as:

$$\beta_i = y_j - y_m = 0 - 1 = -1 \qquad \gamma_i = x_m - x_j = 0 - 2 = -2$$

$$\beta_j = y_m - y_i = 0 - (-1) = 2 \qquad \gamma_j = x_i - x_m = 0 - 0 = 0$$

$$\beta_m = y_i - y_j = -1 - 0 = -1 \qquad \gamma_m = x_j - x_i = 2 - 0 = 2$$

Therefore, the [B] matrix is:

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} = \frac{1}{2(2)}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

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For plane stress conditions, the [D] matrix is:

$$[D] = \frac{30 \times 10^6}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

Substitute the above expressions for [D] and [B] into the general equations for the stiffness matrix:

$$[k] = tA\,[B]^T[D][B]$$

$$k = \frac{(2)30\times 10^6}{4(0.9375)}\begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}\frac{1}{2(2)}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

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Performing the matrix triple product gives:

$$k = 4\times 10^6\begin{bmatrix} 2.5 & 1.25 & -2 & -1.5 & -0.5 & 0.25 \\ 1.25 & 4.375 & -1 & -0.75 & -0.25 & -3.625 \\ -2 & -1 & 4 & 0 & -2 & 1 \\ -1.5 & -0.75 & 0 & 1.5 & 1.5 & -0.75 \\ -0.5 & -0.25 & -2 & 1.5 & 2.5 & -1.25 \\ 0.25 & -3.625 & 1 & -0.75 & -1.25 & 4.375 \end{bmatrix} lb/in$$

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The in-plane stress can be related to displacements by:

$$\{\sigma\} = [D][B]\{d\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30\times 10^6}{0.9375}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}\frac{1}{2(2)}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}\begin{Bmatrix} 0.0 \\ 0.0025\ in \\ 0.0012\ in \\ 0.0 \\ 0.0 \\ 0.0025\ in \end{Bmatrix}$$

The stresses are:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 19{,}200\ psi \\ 4{,}800\ psi \\ -15{,}000\ psi \end{bmatrix}$$

Recall, the relationships for ***principal stresses*** and ***principal angle*** in two-dimensions are:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} = \sigma_{\max}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} = \sigma_{\min}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\left[\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right]$$

Therefore:

$$\sigma_1 = \frac{19{,}200 + 4{,}800}{2} + \sqrt{\left(\frac{19{,}200 - 4{,}800}{2}\right)^2 + (-15{,}000)^2} = 28{,}639\ psi$$

$$\sigma_1 = \frac{19{,}200 + 4{,}800}{2} - \sqrt{\left(\frac{19{,}200 - 4{,}800}{2}\right)^2 + (-15{,}000)^2} = -4{,}639\ psi$$

Treatment of Body and Surface Forces

The general force vector is defined as:

$$\{f\} = \int_V [N]^T \{X\} dV + \{P\} + \int_S [N]^T \{T\} dS$$

Body Force

Let's consider the first term of the above equation.

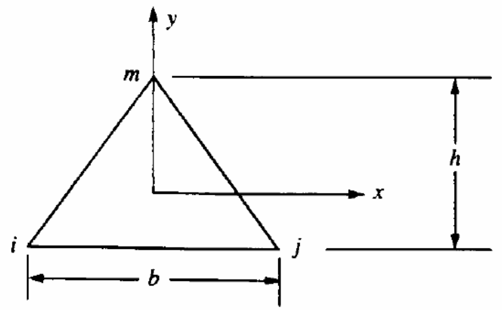
$$\{f_b\} = \int_V [N]^T \{X\} dV$$

where

$$\{X\} = \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix}$$

where X_b and Y_b are the weight densities in the x and y directions, respectively. The force may reflect the effects of gravity, angular velocities, or dynamic inertial forces.

The integration of the $\{f_b\}$ is simplified if the origin of the coordinate system is chosen at the centroid of the element, as shown in the figure below. With the origin placed at the centroid, we can use the definition of a centroid.



$$\int_A x\, dA = 0$$

$$\int_A y\, dA = 0$$

For a given thickness, t, the body force term becomes:

$$\{f_b\} = \int_V [N]^T \{X\} dV = t\int_A [N]^T \{X\} dA$$

Recall the interpolation functions for a place stress/strain triangle:

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y) \qquad N_j = \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y) \qquad N_m = \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)$$

Therefore the terms in the integrand are:

$$\int_A \beta_i x\, dA = \int_A \gamma_i y\, dA = 0$$

and

$$\alpha_i = \alpha_j = \alpha_m = \frac{2A}{3}$$

The body force at node i is given as:

$$\{f_{bi}\} = \frac{tA}{3}\begin{Bmatrix} X_b \\ Y_b \end{Bmatrix}$$

The general body force vector is:

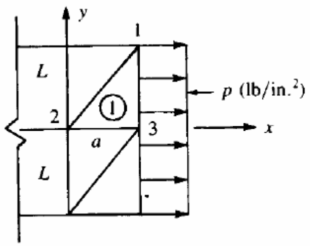
$$\{f_b\} = \begin{Bmatrix} f_{bix} \\ f_{biy} \\ f_{bjx} \\ f_{bjy} \\ f_{bmx} \\ f_{bmy} \end{Bmatrix} = \frac{tA}{3}\begin{Bmatrix} X_b \\ Y_b \\ X_b \\ Y_b \\ X_b \\ Y_b \end{Bmatrix}$$

Surface Force

The third term in the general force vector is defined as:

$$\{f_s\} = \int_S [N]^T \{T\} dS$$

Let's consider the example of a uniform stress p acting between nodes 1 and 3 on the edge of element 1 as shown in figure below.



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The surface traction becomes:

$$\{T\} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{Bmatrix} p \\ 0 \end{Bmatrix}$$

and $[N]^T$ is:

$$[N]^T = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \quad \text{evaluated at } x=a$$

Therefore, the traction force vector is:

$$\{f_s\} = \int_0^t \int_0^L \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} p \\ 0 \end{Bmatrix} dy\, dz \quad \text{evaluated at } x=a$$

After some simplification, the traction force vector is:

$$\{f_s\} = t\int_0^L \begin{bmatrix} N_1 p \\ 0 \\ N_2 p \\ 0 \\ N_3 p \\ 0 \end{bmatrix} dy$$

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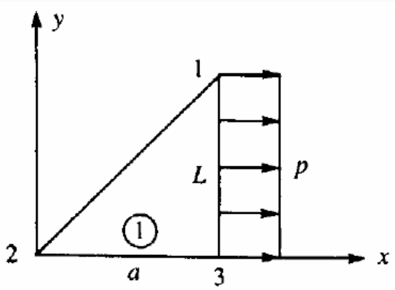
The interpolation function for i = 1 is:

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$

For convenience, let's choose the coordinate system shown in the figure below.



Recall:

$$\alpha_i = x_j y_m - y_j x_m$$

with i = 1, j = 2, and m = 3, we get

$$\alpha_1 = x_2 y_3 - y_2 x_3$$

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If we substitute the coordinates of the triangle show above in the above equation we get:

$$\alpha_1 = 0$$

Similarly, we can find:

$$\beta_1 = 0 \qquad \gamma_1 = a$$

Therefore, the interpolation function, N_1 is:

$$N_1 = \frac{ay}{2A}$$

The remaining interpolation function, N_2 and N_2 are:

$$N_2 = \frac{L(a-x)}{2A} \qquad N_3 = \frac{Lx - ay}{2A}$$

Substituting the interpolation function in the traction force vector expression gives:

$$\{f_s\} = \begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \\ f_{s3x} \\ f_{s3y} \end{Bmatrix} = \frac{pLt}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Explicit Expression for the Constant-Strain Triangle Stiffness Matrix for plain strain case

Recall that: $[k] = tA[B]^T[D][B]$

where $[D]$ for plane strain is:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

Substituting the appropriate definition into the above triple product gives:

$$[k] = \frac{tE}{4A(1+\nu)(1-2\nu)}\begin{bmatrix} \beta_i & 0 & \gamma_i \\ 0 & \gamma_i & \beta_i \\ \beta_j & 0 & \gamma_j \\ 0 & \gamma_j & \beta_j \\ \beta_m & 0 & \gamma_m \\ 0 & \gamma_m & \beta_m \end{bmatrix}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

$$\times\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

Therefore the global stiffness matrix is:

$$\underline{k} = \frac{tE}{4A(1+\nu)(1-2\nu)}$$

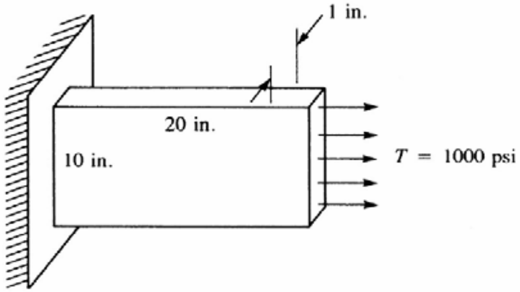
$$\times\begin{bmatrix} \beta_i^2(1-\nu)+\gamma_i^2\left(\frac{1-2\nu}{2}\right) & \beta_i\gamma_i\nu+\beta_i\gamma_i\left(\frac{1-2\nu}{2}\right) & \beta_i\beta_j(1-\nu)+\gamma_i\gamma_j\left(\frac{1-2\nu}{2}\right) \\ & \gamma_i^2(1-\nu)+\beta_i^2\left(\frac{1-2\nu}{2}\right) & \beta_j\gamma_i\nu+\beta_i\gamma_j\left(\frac{1-2\nu}{2}\right) \\ & & \beta_j^2(1-\nu)+\gamma_j^2\left(\frac{1-2\nu}{2}\right) \\ \text{Symmetry} & & \end{bmatrix}$$

$$\begin{bmatrix} \beta_i\gamma_j\nu+\beta_j\gamma_i\left(\frac{1-2\nu}{2}\right) & \beta_i\beta_m(1-\nu)+\gamma_i\gamma_m\left(\frac{1-2\nu}{2}\right) & \beta_i\gamma_m\nu+\beta_m\gamma_i\left(\frac{1-2\nu}{2}\right) \\ \gamma_i\gamma_j(1-\nu)+\beta_i\beta_j\left(\frac{1-2\nu}{2}\right) & \beta_m\gamma_i\nu+\beta_i\gamma_m\left(\frac{1-2\nu}{2}\right) & \gamma_i\gamma_m(1-\nu)+\beta_i\beta_m\left(\frac{1-2\nu}{2}\right) \\ \beta_j\gamma_j\nu+\beta_j\gamma_j\left(\frac{1-2\nu}{2}\right) & \beta_j\beta_m(1-\nu)+\gamma_j\gamma_m\left(\frac{1-2\nu}{2}\right) & \beta_j\gamma_m\nu+\gamma_j\beta_m\left(\frac{1-2\nu}{2}\right) \\ \gamma_j^2(1-\nu)+\beta_j^2\left(\frac{1-2\nu}{2}\right) & \beta_m\gamma_j\nu+\beta_j\gamma_m\left(\frac{1-2\nu}{2}\right) & \gamma_j\gamma_m(1-\nu)+\beta_j\beta_m\left(\frac{1-2\nu}{2}\right) \\ & \beta_m^2(1-\nu)+\gamma_m^2\left(\frac{1-2\nu}{2}\right) & \gamma_m\beta_m\nu+\beta_m\gamma_m\left(\frac{1-2\nu}{2}\right) \\ & & \gamma_m^2(1-\nu)+\beta_m^2\left(\frac{1-2\nu}{2}\right) \end{bmatrix}$$

The stiffness matrix is a function of the global coordinates x and y, the material properties, the thickness and area of the element

FINITE ELEMENT SOLUTION OF A PLANE STRESS EXAMPLE

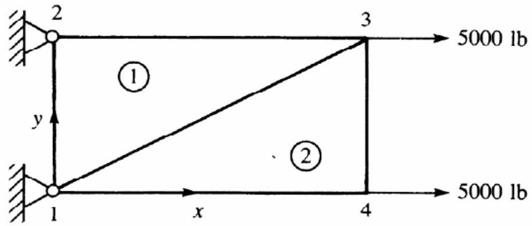
Consider the thin plate subjected to the surface traction shown in the figure below.



Assume plane stress conditions. Let $E = 30 \times 10^6$ psi, $\nu = 0.30$, and $t = 1$ in. Determine the nodal displacements and the element stresses.

Discretization

Let's discretize the plate into two elements as shown below:



This level of discretization will probably not yield practical results for displacement and stresses: however; it is useful example for a longhand solution.

The tensile traction forces can be converted into nodal forces as follows:

$$\{f_s\} = \begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \\ f_{s3x} \\ f_{s3y} \end{Bmatrix} = \frac{pLt}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1{,}000\ psi(1\ in)10\ in}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5{,}000\ lb \\ 0 \\ 0 \\ 0 \\ 5{,}000\ lb \\ 0 \end{bmatrix}$$

The governing global matrix equations are:

$$\{F\} = [K]\{d\}$$

Expanding the above matrices gives:

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 5{,}000\ lb \\ 0 \\ 5{,}000\ lb \\ 0 \end{Bmatrix} = [K]\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix} = [K]\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

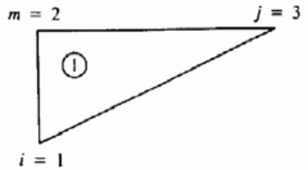
where $[K]$ is an 8 x 8 matrix before deleting the rows and columns accounting for the boundary supports.

Assemblage of the Stiffness Matrix

The global stiffness matrix is assembled by superposition of the individual element stiffness matrices. The element stiffness matrix is:

$$[k] = tA[B]^T[D][B]$$

For **element 1**: the coordinates are $x_i = 0$, $y_i = 0$, $x_j = 20$, $y_j = 10$, $x_m = 0$, and $y_m = 10$. The area of the triangle is:



$$A = \frac{bh}{2}$$

$$A = \frac{(20)(10)}{2} = 100\ in.^2$$

The matrix [*B*] is:

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

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We need to calculate the element β's and γ's as:

$$\beta_i = y_j - y_m = 10 - 10 = 0 \qquad \gamma_i = x_m - x_j = 0 - 20 = -20$$

$$\beta_j = y_m - y_1 = 10 - 0 = 10 \qquad \gamma_j = x_i - x_m = 0 - 0 = 0$$

$$\beta_m = y_i - y_j = 0 - 10 = -10 \qquad \gamma_m = x_i - x_j = 20 - 0 = 20$$

Therefore, the [*B*] matrix is:

$$[B] = \frac{1}{200}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix} {}^1\!/_{in}$$

For plane stress conditions, the [*D*] matrix is:

$$[D] = \frac{30 \times 10^6}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} psi$$

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Therefore:

$$[B]^T[D] = \frac{30(10^6)}{200(0.91)}\begin{bmatrix} 0 & 0 & -20 \\ 0 & -20 & 0 \\ 10 & 0 & 0 \\ 0 & 0 & 10 \\ -10 & 0 & 20 \\ 0 & 20 & -10 \end{bmatrix}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Simplifying the above expression gives:

$$[B]^T[D] = \frac{30(10^6)}{200(0.91)}\begin{bmatrix} 0 & 0 & -7 \\ -6 & -20 & 0 \\ 10 & 3 & 0 \\ 0 & 0 & 3.5 \\ -10 & -3 & 7 \\ 6 & 20 & -3.5 \end{bmatrix}$$

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The element stiffness matrix is:

$$[k] = tA[B]^T[D][B]$$

therefore:

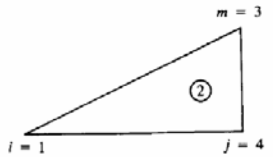
$$tA[B]^T[D][B] = 1(100)\frac{(0.15)(10^6)}{0.91}\begin{bmatrix} 0 & 0 & -7 \\ -6 & -20 & 0 \\ 10 & 3 & 0 \\ 0 & 0 & 3.5 \\ -10 & -3 & 7 \\ 6 & 20 & -3.5 \end{bmatrix}$$

$$\times \frac{1}{200}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix}$$

Simplifying the above expression gives:

$$[k] = \frac{75{,}000}{0.91}\begin{array}{c} \begin{matrix} u_1 & v_1 & u_3 & v_3 & u_2 & v_2 \end{matrix} \\ \begin{bmatrix} 140 & 0 & 0 & -70 & -140 & 70 \\ 0 & -400 & -60 & 0 & 60 & -400 \\ 0 & -60 & 100 & 0 & -100 & 60 \\ -70 & 0 & 0 & 35 & 70 & -35 \\ -140 & 60 & -100 & 70 & 240 & -130 \\ 70 & -400 & 60 & -35 & -130 & 435 \end{bmatrix} \end{array}$$

For **element 2**: the coordinates are $x_i = 0$, $y_i = 0$, $x_j = 20$, $y_j = 0$, $x_m = 20$, and $y_m = 10$. The area of the triangle is:



$$A = \frac{(20)(10)}{2} = 100 \ in.^2$$

We need to calculate the element β's and γ's as:

$$\beta_i = y_j - y_m = 0 - 10 = -10 \qquad \gamma_i = x_m - x_j = 20 - 20 = 0$$

$$\beta_j = y_m - y_1 = 10 - 0 = 10 \qquad \gamma_j = x_i - x_m = 0 - 20 = -20$$

$$\beta_m = y_i - y_j = 0 - 0 = 0 \qquad \gamma_m = x_i - x_j = 20 - 0 = 20$$

Therefore, the [B] matrix is:

$$[B] = \frac{1}{200}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix} \frac{1}{in}$$

For plane stress conditions, the [D] matrix is:

$$[D] = \frac{30 \times 10^6}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} psi$$

Therefore:

$$[B]^T[D] = \frac{30(10^6)}{200(0.91)}\begin{bmatrix} -10 & 0 & 0 \\ 0 & 0 & -10 \\ 10 & 0 & -20 \\ 0 & -20 & 10 \\ 0 & 0 & 20 \\ 0 & 20 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Simplifying the above expression gives:

$$[B]^T[D] = \frac{30(10^6)}{200(0.91)}\begin{bmatrix} -10 & -3 & 0 \\ 0 & 0 & -3.5 \\ 10 & 3 & -7 \\ 0 & -20 & 3.5 \\ -6 & 0 & 7 \\ 6 & 20 & 0 \end{bmatrix}$$

The element stiffness matrix is:

$$[k] = tA[B]^T[D][B]$$

therefore:

$$tA[B]^T[D][B] = 1(100)\frac{(0.15)(10^6)}{0.91}\begin{bmatrix} -10 & -3 & 0 \\ 0 & 0 & -3.5 \\ 10 & 3 & -7 \\ 0 & -20 & 3.5 \\ -6 & 0 & 7 \\ 6 & 20 & 0 \end{bmatrix} \times \frac{1}{200}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix}$$

Simplifying the above expression gives:

$$[k] = \frac{75{,}000}{0.91}\begin{bmatrix} 100 & 0 & -100 & 60 & 0 & -60 \\ 0 & 35 & 70 & -35 & -70 & 0 \\ -100 & 70 & 240 & -130 & -140 & 60 \\ 60 & -35 & -130 & 435 & 70 & -400 \\ 0 & -70 & -140 & 70 & 140 & 0 \\ -60 & 0 & 60 & -400 & 0 & 400 \end{bmatrix}$$

(columns: u_1, v_1, u_4, v_4, u_3, v_3)

Element 1:

(columns: u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4)

$$[k] = \frac{375{,}000}{0.91}\begin{bmatrix} 28 & 0 & -28 & 14 & 0 & -14 & 0 & 0 \\ 0 & 80 & 12 & -80 & -12 & 0 & 0 & 0 \\ -28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\ 14 & -80 & -26 & 87 & 12 & -7 & 0 & 0 \\ 0 & -12 & -20 & 12 & 20 & 0 & 0 & 0 \\ -14 & 0 & 14 & -7 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2:

(columns: u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4)

$$[k] = \frac{375{,}000}{0.91}\begin{bmatrix} 20 & 0 & 0 & 0 & 0 & -12 & -20 & 12 \\ 0 & 7 & 0 & 0 & -14 & 0 & 14 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -14 & 0 & 0 & 28 & 0 & -28 & 14 \\ -12 & 0 & 0 & 0 & 0 & 80 & 12 & -80 \\ -20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\ 12 & -7 & 0 & 0 & 14 & -80 & -26 & 87 \end{bmatrix}$$

Using the superposition, the total global stiffness matrix is:

$$
\begin{array}{c} \\ [k] = \dfrac{375{,}000}{0.91} \end{array}
\begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array}
$$

$$
[k] = \frac{375{,}000}{0.91}\begin{bmatrix}
48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\
0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\
-28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\
14 & 80 & -26 & 87 & 12 & -7 & 0 & 0 \\
0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\
-26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\
-20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\
12 & -7 & 0 & 0 & 14 & -80 & -26 & 87
\end{bmatrix}
$$

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The governing global matrix equations are:

$$\{F\} = [K]\{d\}$$

$$
\begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 5{,}000\ lb \\ 0 \\ 500\ lb \\ 0 \end{Bmatrix}
= \frac{375{,}000}{0.91}\begin{bmatrix}
48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\
0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\
-28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\
14 & 80 & -26 & 87 & 12 & -7 & 0 & 0 \\
0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\
-26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\
-20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\
12 & -7 & 0 & 0 & 14 & -80 & -26 & 87
\end{bmatrix}
\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}
$$

Applying the boundary conditions:

$$d_{1x} = d_{1y} = d_{2x} = d_{2y} = 0$$

The governing equations are:

$$
\begin{Bmatrix} 5{,}000\ lb \\ 0 \\ 5{,}000\ lb \\ 0 \end{Bmatrix}
= \frac{375{,}000}{0.91}\begin{bmatrix}
48 & 0 & -28 & 14 \\
0 & 87 & 12 & -80 \\
-28 & 12 & 48 & -26 \\
14 & -80 & -26 & 87
\end{bmatrix}
\begin{Bmatrix} d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}
$$

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Solving the equations gives:

$$
\begin{Bmatrix} d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}
= (10^{-6})\begin{Bmatrix} 609.6 \\ 4.2 \\ 663.7 \\ 104.1 \end{Bmatrix} in
$$

The exact solution for the displacement at the free end of the one-dimensional bar subjected to a tensile force is:

$$\delta = \frac{PL}{AE} = \frac{(10{,}000)20}{10(30\times 10^{6})} = 670\times 10^{-6}\ in$$

The in-plane stress can be related to displacements by:

$$\{s\} = [D][B]\{d\}$$

$$
\{\sigma\} = \frac{E}{2A(1-\nu^2)}
\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}
\begin{bmatrix}
\beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\
0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\
\gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m
\end{bmatrix}
\begin{Bmatrix} d_{ix} \\ d_{iy} \\ d_{jx} \\ d_{jy} \\ d_{mx} \\ d_{my} \end{Bmatrix}
$$

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Element 1:

$$
\{\sigma\} = \frac{E}{2A(1-\nu^2)}
\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}
\begin{bmatrix}
\beta_1 & 0 & \beta_3 & 0 & \beta_2 & 0 \\
0 & \gamma_1 & 0 & \gamma_3 & 0 & \gamma_2 \\
\gamma_1 & \beta_1 & \gamma_3 & \beta_3 & \gamma_2 & \beta_2
\end{bmatrix}
\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{3x} \\ d_{3y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}
$$

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}
= \frac{30(10^{6})(10^{-6})}{0.96(200)}
\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}
\begin{bmatrix}
0 & 0 & 10 & 0 & -10 & 0 \\
0 & -20 & 0 & 0 & 0 & 20 \\
-20 & 0 & 0 & 10 & 20 & -10
\end{bmatrix}
\begin{Bmatrix} 0.0 \\ 0.0 \\ 609.6 \\ 4.2 \\ 0.0 \\ 0.0 \end{Bmatrix}
$$

The stresses are:

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}
= \begin{bmatrix} 1{,}005\ psi \\ 301\ psi \\ 2.4\ psi \end{bmatrix}
$$

Element 2:

$$\{\sigma\} = \frac{E}{2A(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}\begin{bmatrix} \beta_1 & 0 & \beta_4 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_4 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_4 & \beta_4 & \gamma_3 & \beta_3 \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{4x} \\ d_{4y} \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30(10^6)(10^{-6})}{0.96(200)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{bmatrix} 10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & 10 & -20 & 10 & 20 & 0 \end{bmatrix}\begin{Bmatrix} 0.0 \\ 0.0 \\ 663.7 \\ 104.1 \\ 609.6 \\ 4.2 \end{Bmatrix}$$

The stresses are:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 995\ psi \\ -1.2\ psi \\ -2.4\ psi \end{bmatrix}$$

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The ***principal*** stresses and principal angle are:

$$s_1 = \frac{995-1.2}{2} + \sqrt{\left(\frac{995+1.2}{2}\right)^2 + (-2.4)^2} = 995\ psi$$

$$\sigma_2 = \frac{995-1.2}{2} - \sqrt{\left(\frac{995+1.2}{2}\right)^2 + (-2.4)^2} = -1.1\ psi$$

$$\theta_p = \frac{1}{2}tan^{-1}\left[\frac{2(-2.4)}{995+1.2}\right] \approx 0^\circ$$