ADVANCED MECHANICS OF MATERIALS

By

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INSTITUTE OF ENGINEERING

SURANAREE UNIVERSITY OF TECHNOLOGY

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SURANAREE UNIVERSITY OF TECHNOLOGY
INSTITUTE OF ENGINEERING
SCHOOL OF CIVIL ENGINEERING

410 611 ADVANCED MECHANICS OF MATERIALS 1st Trimester /2002

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Prerequisite: 410 212 Mechanics of Materials II or consent of instructor

Objectives: Students successfully completing this course will

1. understand the concept of fundamental theories of the advanced mechanics of material;
2. be able to simplify a complex mechanic problem down to one that can be analyzed;
3. understand the significance of the solution to the problem of any assumptions made.

Textbooks:

1. **Advanced Mechanics of Materials**; 4th Edition, A.P. Boresi and O.M. Sidebottom, John Wiley & Sons, 1985
2. **Advanced Mechanics of Materials**; 2nd Edition, R.D. Cook and W.C. Young, Prentice Hall, 1999
3. **Theory of Elastic Stability**; 2nd Edition, S.P. Timoshenko and J.M. Gere, McGraw-Hill, 1963
4. **Theory of Elasticity**; 3rd Edition, S.P. Timoshenko and J.N. Goodier, McGraw-Hill, 1970
5. **Theory of Plates and Shells**; 2nd Edition, S.P. Timoshenko and S. Woinowsky-Krieger, McGraw-Hill, 1970
6. **Mechanical Behavior of Materials**; 2nd Edition, N.E. Dowling, Prentice Hall, 1999
7. **Mechanics of Materials**; 3th Edition, R.C. Hibbeler, Prentice Hall, 1997

Course Contents:

Chapters	Topics
1	Theories of Stress and Strain
2	Stress-Strain Relations
3	Elements of Theory of Elasticity
4	Applications of Energy Methods
5	Static Failure and Failure Criteria
6	Fatigue
7	Introduction to Fracture Mechanics
8	Beams on Elastic Foundation
9	Plate Bending
10	Buckling and Instability

Conduct of Course:

Homework, Quizzes, and Projects	30%
Midterm Examination	35%
Final Examination	35%

Grading Guides:

90 and above	A
85-89	B+
80-84	B
75-79	C+
70-74	C

65-69	D+
60-64	D
below 60	F

The above criteria may be changed at the instructor’s discretion.

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Chapter 1

Theories of Stress and Strain

1.1 Definition of Stress at a Point

Mechanics of materials is a branch of mechanics that studies

1.) The relationships between the external loads applied to a deformable body and intensity of internal forces acting within the body.

2.) The deformation and the stability of the body under the action of the external loads.

In this study, the body is made of the material that is continuous (consist of a continuum or uniform distribution of matter having no voids) and cohesive (all portions of the material are connected together without breaks, cracks, and separations).

The external loads that we are interested can be idealized as concentrated force, surface force, and linear distributed load as shown in Fig. 1.1.

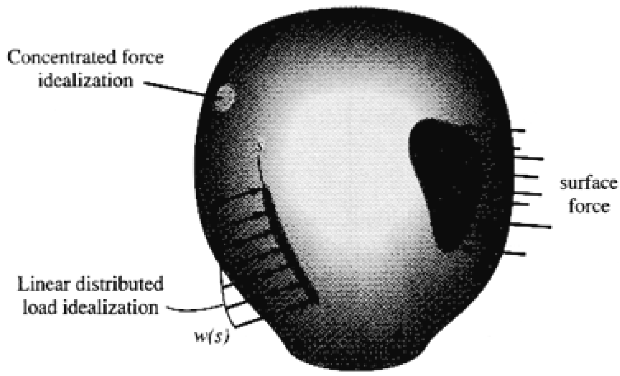


Fig. 1.1

Consider the body subjected to a loading condition as shown in Fig. 1.2a. Under the action of these loads, the body is deformed and has internal forces to hold the body together. At a given section, the internal forces are distributed as shown in Fig. 1.2b and we can find the resultant of the internal forces at a given point O as shown in Fig. 1.2c.

The distribution of the internal forces at a given point on the sectioned area of the body can be determined by using the concept of stress.

Stress describes the intensity of the internal force on a specific plane or area passing through a point as shown in Fig. 1.3. It can be classified into two types based on its acting directions: normal stress and shear stress. Since the stresses generally vary from point to point, the definitions of stresses must relate to an infinitesimal element.

Normal stress or σ is the intensity of force that acts normal to the area ΔA. If the normal force or stress *pulls* on the area element ΔA, it is referred to as *tensile stress,* whereas if it pushes on the area ΔA, it is referred to as *compressive stress.* It can be defined as

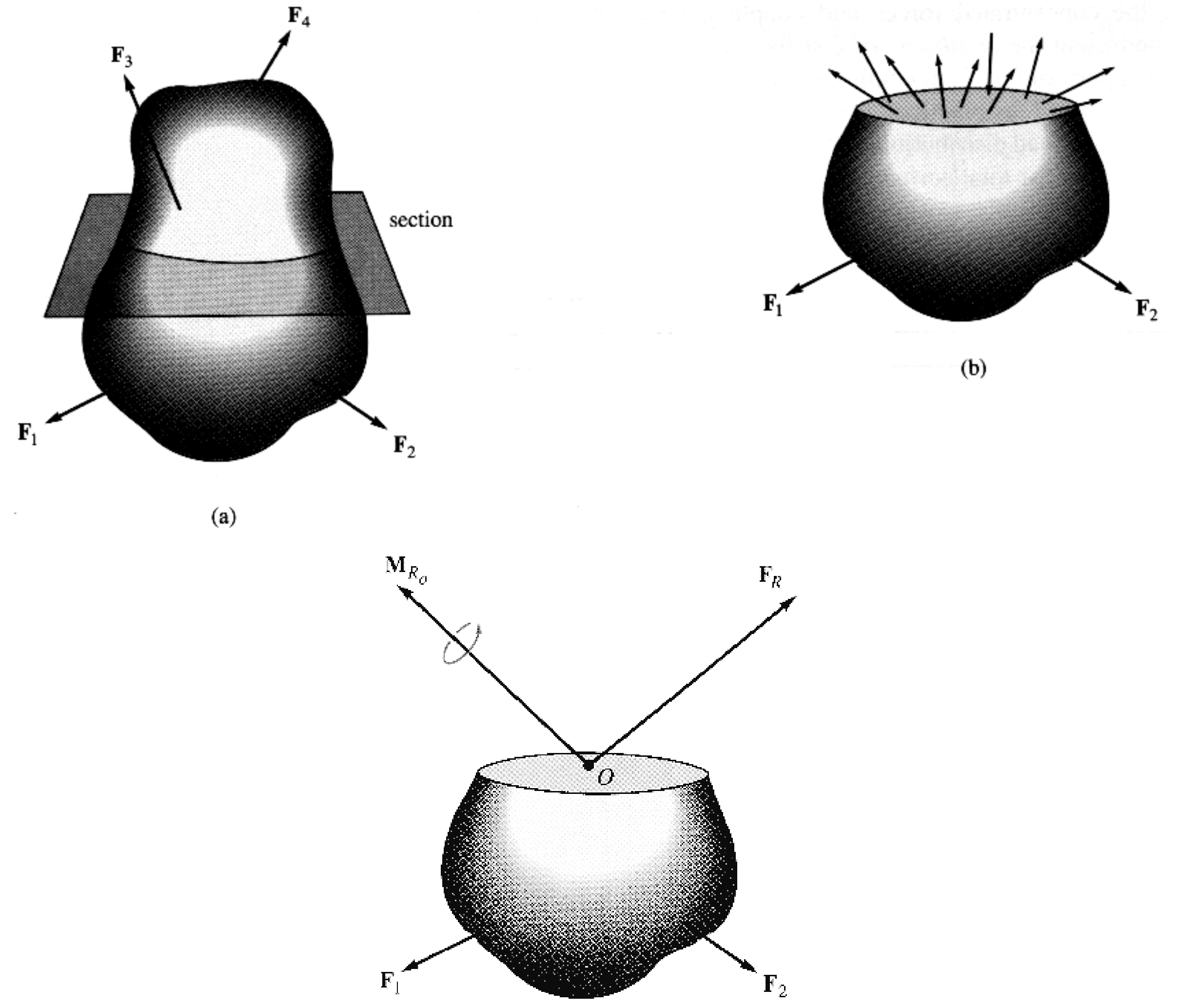


Fig. 1.2

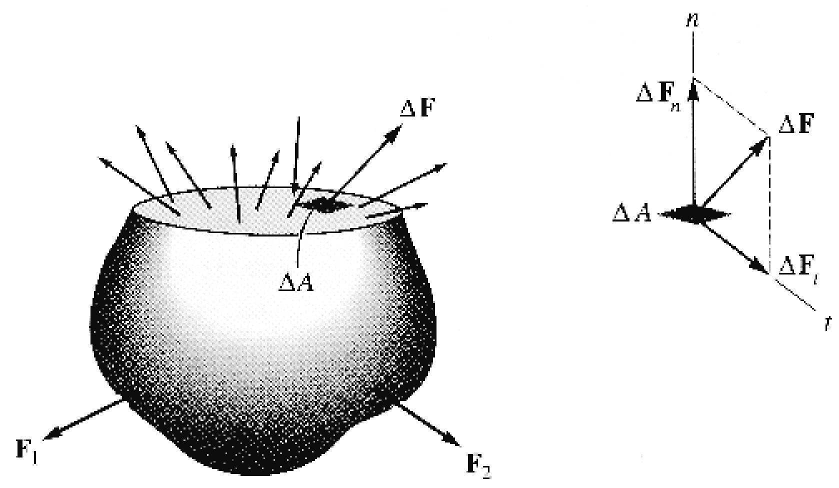


Fig. 1.3

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A}$$

Shear stress or τ is the intensity of force that acts tangent to the area ΔA.

$$\tau = \lim_{\Delta A \to 0} \frac{\Delta F_t}{\Delta A}$$

1.2 Stress Notation

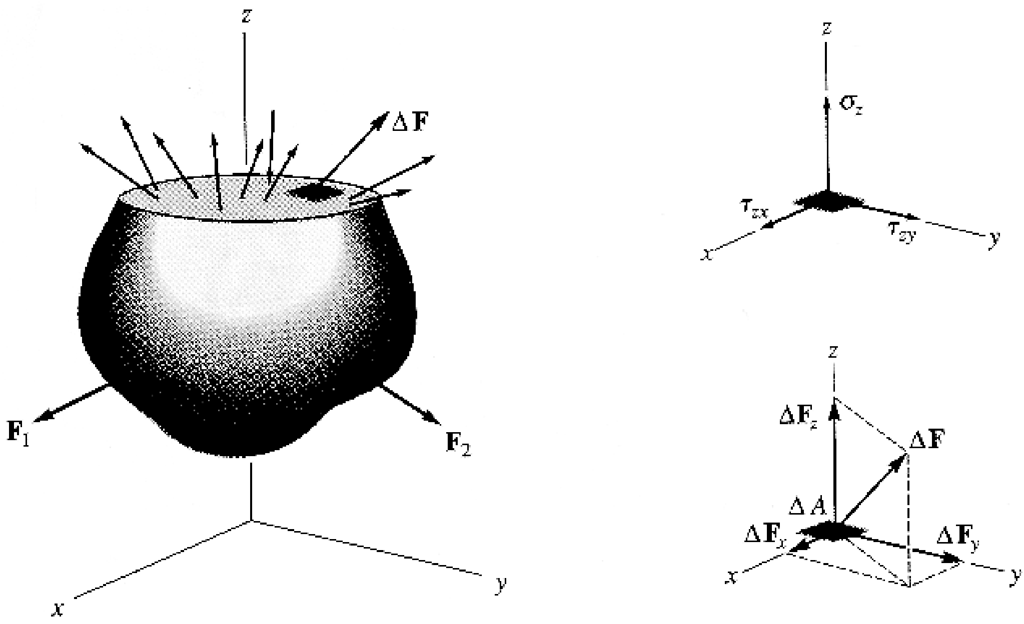


Fig. 1.4

By passing the imaginary section through the body parallel to the *x-y* plane as shown in Fig. 1.4, the stress on the element area $\Delta A = \Delta x \Delta y$ can be resolved into stress components in the rectangular orthogonal Cartesian coordinate axes *x*, *y*, and *z* as σ_z, τ_{zx}, and τ_{zy} where

$$\sigma_z = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A}$$

$$\tau_{zx} = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}$$

$$\tau_{zy} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}$$

Notation

- The first subscript notation refers to the orientation of area ΔA, which is perpendicular to the subscript notation.
- The second subscript notation refers to the direction line of the stress.

By passing the imaginary section through the body parallel to the *x-z* plane as shown in Fig. 1.5, we obtain the stress components as σ_y, τ_{yx}, τ_{yz}.

By passing the imaginary section through the body parallel to the *y-z* plane as shown in Fig. 1.6, we obtain the stress components as σ_x, τ_{xy}, τ_{xz}.

If we continue cut the body in this manner by using the corresponding parallel plane, we will obtain a cubic volume element of material that represents the state of stress acting around the chosen point on the body as shown in Fig. 1.7.

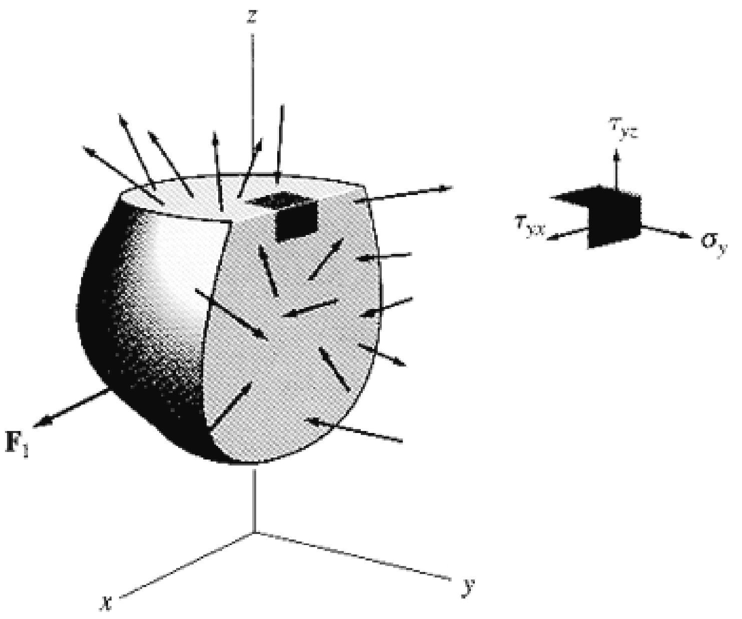


Fig. 1.5

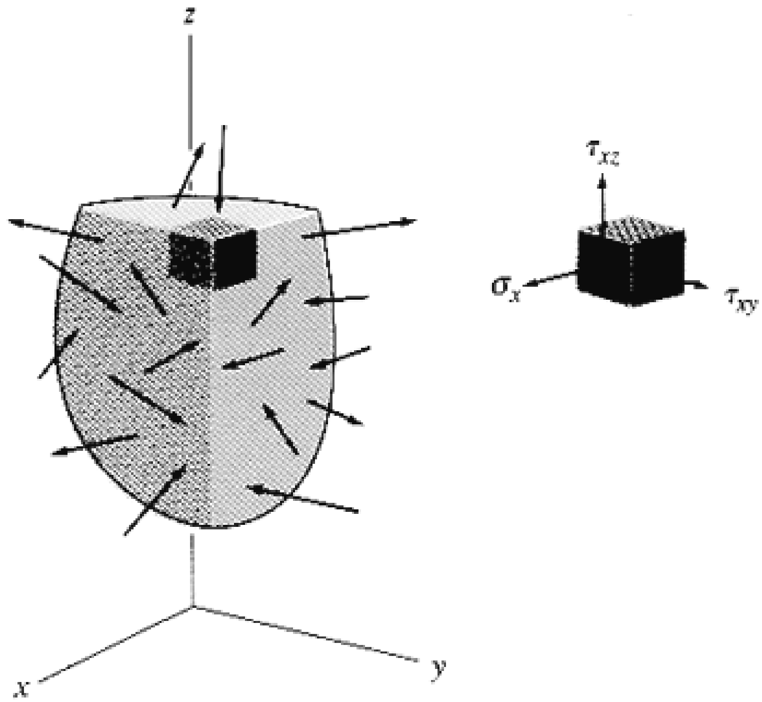


Fig. 1.6

In array form, we have

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

Stress on the plane perpendicular to x-axis

Stresses in the x-direction

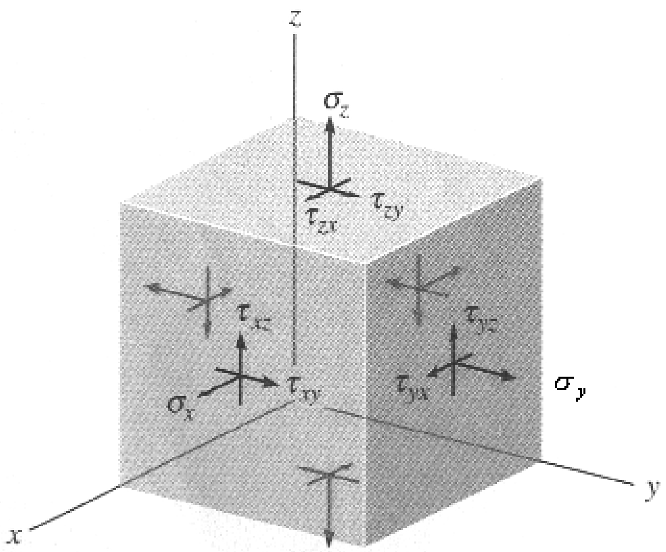


Fig. 1.7

1.3 Symmetry of the Stress Array and Stress on an Arbitrarily Oriented Plane

If the stress around the chosen point on the body is constant, some of the stress components can be reduced by using the force and moment equilibrium for the element.

Normal Stress Components

For a constant state of stress as shown in Fig. 1.8, each of the three normal stress components must be equal in magnitude, but opposite in direction.

$\sum F_x = 0$; $$\sigma_x(\Delta y \Delta z) - \sigma'_x(\Delta y \Delta z) = 0$$

$$\sigma_x = \sigma'_x$$

Similarly, we have

$\sum F_y = 0$ $$\sigma_y = \sigma'_y$$

$\sum F_z = 0$; $$\sigma_z = \sigma'_z$$

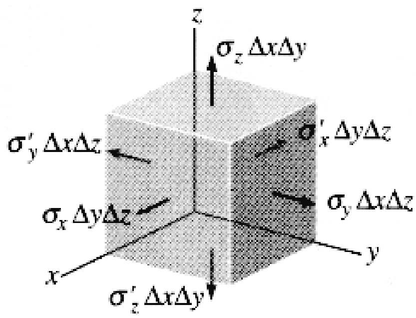


Fig. 1.8

Shear Stress Components

For a constant state of stress as shown in Fig. 1.9, pairs of shear stresses on adjacent faces of the element must have equal magnitude and be directed either toward or away from the corners of the element.

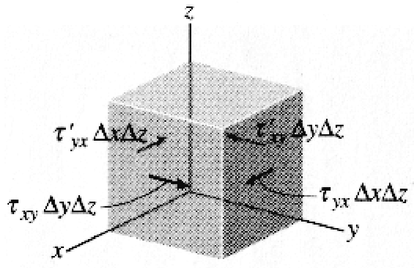


Fig. 1.9

$\sum F_x = 0$; $$\tau_{yx}(\Delta x\Delta z) - \tau'_{yx}(\Delta x\Delta z) = 0$$

$$\tau_{yx} = \tau'_{yx}$$

$\sum M_z = 0$; $$\tau_{xy}(\Delta y\Delta z)\Delta x - \tau_{yx}(\Delta x\Delta z)\Delta y = 0$$

$$\tau_{xy} = \tau_{yx}$$

Similarly, we have

$$\tau_{yz} = \tau'_{yz} = \tau_{zy} = \tau'_{zy}$$

$$\tau_{xz} = \tau'_{xz} = \tau_{zx} = \tau'_{zx}$$

Therefore, in matrix form,

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ \text{Sym.} & & \sigma_z \end{bmatrix}$$

Stress Acting on Arbitrary Plane

The stress vector on the plane that is perpendicular to the x-axis as shown in Fig. 1.10 can be written as

$$\vec{\sigma}_x = \sigma_x\hat{i} + \tau_{xy}\hat{j} + \tau_{xz}\hat{k}$$

In a similar fashion, the stress vector on the planes that are perpendicular to the y- and z-axes can be written as

$$\vec{\sigma}_y = \tau_{yx}\hat{i} + \sigma_y\hat{j} + \tau_{yz}\hat{k}$$

$$\vec{\sigma}_z = \tau_{zx}\hat{i} + \tau_{zy}\hat{j} + \sigma_z\hat{k}$$

Consider an arbitrary oblique plane P as shown in Fig. 1.11. Let the plane is defined by a unit normal vector

$$\vec{N} = l\hat{i} + m\hat{j} + n\hat{k}$$

where $l = \cos\alpha$, $m = \cos\beta$, and $n = \cos\gamma$. From Fig. 1.12, we have $l^2 + n^2 = d^2$ and $d^2 + m^2 = 1$. Thus,

$$l^2 + m^2 + n^2 = 1$$

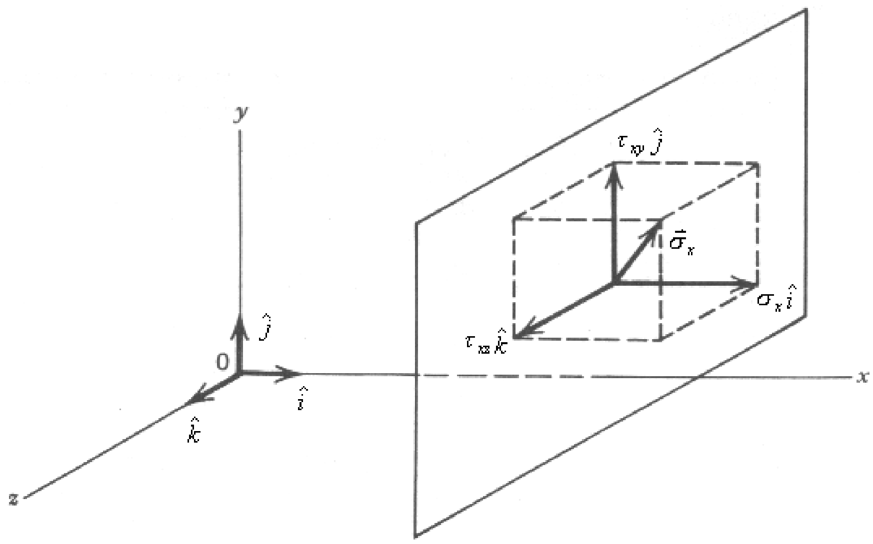


Fig. 1.10

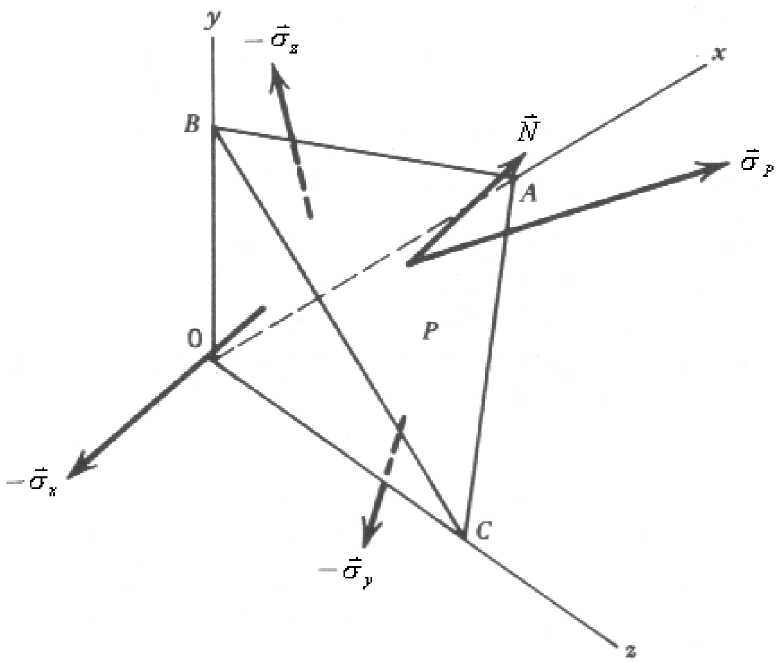


Fig. 1.11

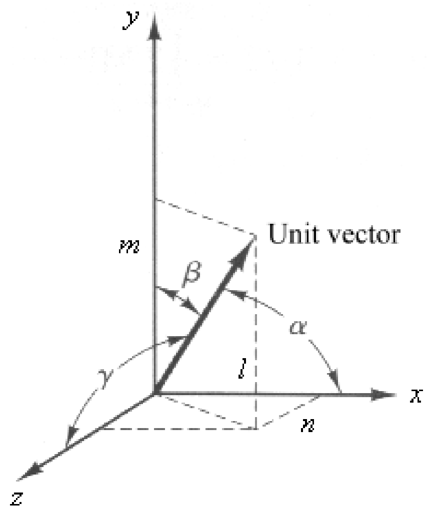


Fig. 1.12

In addition, if the infinitesimal area ABC is defines as dA_{ABC},

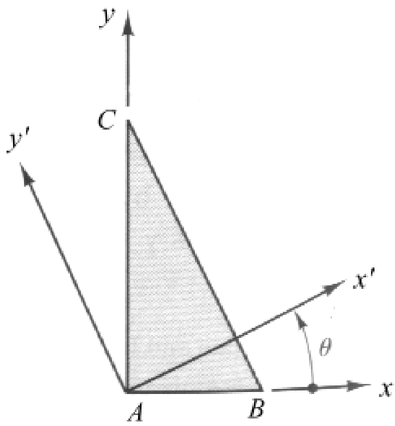
$$dA_{0BC} = l dA_{ABC}$$

$$dA_{0AC} = m dA_{ABC}$$

$$dA_{0AB} = n dA_{ABC}$$

Remark:

The area dA_{0BC}, dA_{0AC}, and dA_{0AB} are the projection of area dA_{ABC} on the respective coordinate plane. Let consider the figure of the wedge shown below and compare the volumes determined from two methods.



1. Let the area associated with the side AC be A_{AC} and corresponding wedge height be AB. The volume of the wedge is

$$\frac{1}{2} AB(A_{AC})$$

2. Let the area associated with the side CB be A_{CB} and corresponding wedge height be $AB\cos\theta = ABl$. The volume of the wedge is

$$\frac{1}{2} ABl(A_{CB})$$

By equating the two volumes, we have

$$A_{AC} = l A_{CB}$$

$$l = \frac{A_{AC}}{A_{CB}}$$

Thus, we can write the stress vector $\vec{\sigma}_P$ on the oblique plane P by summing the force vectors acting on the stress element as shown in Fig. 1.11.

$$\vec{\sigma}_P (dA_{ABC}) = \vec{\sigma}_x (dA_{0BC}) + \vec{\sigma}_y (dA_{0AC}) + \vec{\sigma}_z (dA_{0AB})$$

$$\vec{\sigma}_P = \vec{\sigma}_x l + \vec{\sigma}_y m + \vec{\sigma}_z n$$

$$\begin{aligned}\vec{\sigma}_P = &(\sigma_x \hat{i} + \tau_{xy} \hat{j} + \tau_{xz} \hat{k})l + \\ &(\tau_{yx} \hat{i} + \sigma_y \hat{j} + \tau_{yz} \hat{k})m + \\ &(\tau_{zx} \hat{i} + \tau_{zy} \hat{j} + \sigma_z \hat{k})n\end{aligned}$$

$$\begin{aligned}\vec{\sigma}_P = &(\sigma_x l + \tau_{yx} m + \tau_{zx} n)\hat{i} + \\ &(\tau_{xy} l + \sigma_y m + \tau_{zy} n)\hat{j} + \\ &(\tau_{xz} l + \tau_{yz} m + \sigma_z n)\hat{k}\end{aligned}$$

Also, the projections of the stress vector $\vec{\sigma}_P$ on the x, y, and z-axes may be written as

$$\vec{\sigma}_P = \sigma_{Px} \hat{i} + \sigma_{Py} \hat{j} + \sigma_{Pz} \hat{k}$$

Comparing the stress vector $\vec{\sigma}_P$, we have

$$\sigma_{Px} = \sigma_x l + \tau_{yx} m + \tau_{zx} n$$

$$\sigma_{Py} = \tau_{xy} l + \sigma_y m + \tau_{zy} n$$

$$\sigma_{Pz} = \tau_{xz} l + \tau_{yz} m + \sigma_z n$$

Normal Stress and Shearing Stress on an Oblique Plane

The normal stress on the plane P or σ_{PN} is the projection of the stress vector $\vec{\sigma}_P$ in the direction of the unit normal vector $\vec{N}$. Thus, the magnitude of the normal stress σ_{PN} can be determined from

$$\begin{aligned}\sigma_{PN} &= \vec{\sigma}_P \cdot \vec{N} \\ &= (\sigma_{Px} \hat{i} + \sigma_{Py} \hat{j} + \sigma_{Pz} \hat{k}) \cdot (l\hat{i} + m\hat{j} + n\hat{k}) \\ &= \sigma_{Px} l + \sigma_{Py} m + \sigma_{Pz} n\end{aligned}$$

Since $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, and $\tau_{xz} = \tau_{zx}$, we have

$$\begin{aligned}\sigma_{PN} &= (\sigma_x l + \tau_{xy} m + \tau_{xz} n)l + (\tau_{xy} l + \sigma_y m + \tau_{yz} n)m + (\tau_{xz} l + \tau_{yz} m + \sigma_z n)n \\ &= \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2mn\tau_{yz} + 2nl\tau_{xz} + 2lm\tau_{xy}\end{aligned}$$

In matrix notation, we can see that if we write the unit normal vector $\vec{N} = l\hat{i} + m\hat{j} + n\hat{k}$ in the form of

$$\vec{N} = \lfloor l \quad m \quad n \rfloor \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix}$$

or,

$$\vec{N} = \lfloor N \rfloor \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix}$$

Then, we have

$$\sigma_{PN} = \lfloor l \quad m \quad n \rfloor \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$$

or,

$$\sigma_{PN} = \lfloor N \rfloor [\sigma] \lfloor N \rfloor^T$$

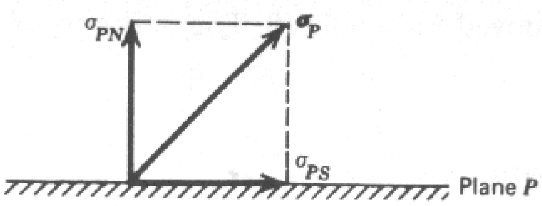


Fig. 1.13

Also, form Fig. 1.13, the magnitude of the shearing stress on plane P or σ_{PS} can be determined from the equation

$$\sigma_{PS} = \sqrt{\sigma_P^2 - \sigma_{PN}^2} = \sqrt{\sigma_{Px}^2 + \sigma_{Py}^2 + \sigma_{Pz}^2 - \sigma_{PN}^2}$$

Of all the infinite number of planes passing through point O, there is a set of three mutually perpendicular planes that the normal stress σ_{PN} has a maximum value called the ***principal planes***. The maximum normal stress is called the ***maximum principal stress***. On these planes, the shearing stresses vanish. Also, the three mutually perpendicular axes that are normal to the three planes are called ***principal axes***.

1.4 Transformation of Stress, Principal Stresses, and Other Properties

Transformation of Stress

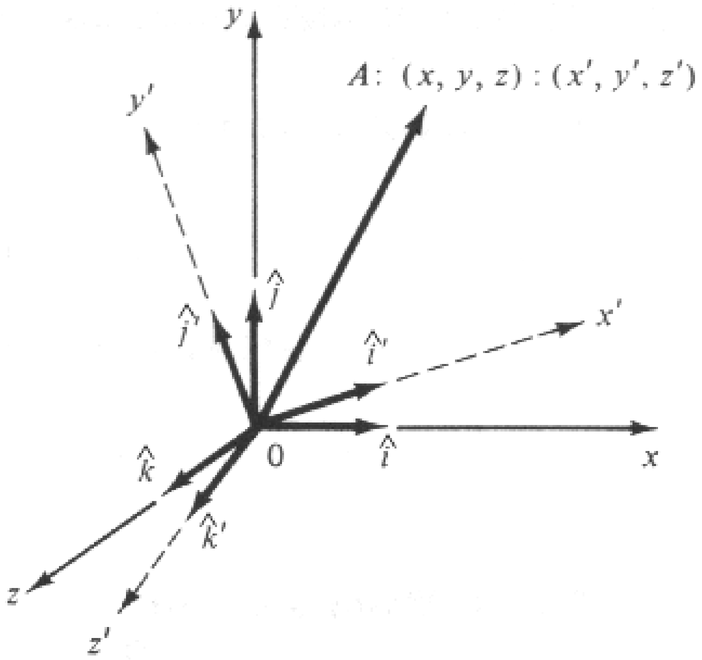


Fig. 1.14

Let (x, y, z) and (x', y', z') denote two rectangular coordinate systems with a common origin as shown in Fig. 1.14. Also, let a general point in space A has coordinate (x, y, z) and (x', y', z') in the respective coordinate system. The direction cosines between the coordinate axes (x, y, z) and (x', y', z') can be determined by finding the coordinate x', y', z' of the point A in term of the coordinate x, y, z.

$$\vec{R} = O\vec{A} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{Bmatrix} \vec{R}\cdot\hat{i}' \\ \vec{R}\cdot\hat{j}' \\ \vec{R}\cdot\hat{k}' \end{Bmatrix} = \begin{bmatrix} \hat{i}\cdot\hat{i}' & \hat{j}\cdot\hat{i}' & \hat{k}\cdot\hat{i}' \\ \hat{i}\cdot\hat{j}' & \hat{j}\cdot\hat{j}' & \hat{k}\cdot\hat{j}' \\ \hat{i}\cdot\hat{k}' & \hat{j}\cdot\hat{k}' & \hat{k}\cdot\hat{k}' \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = [T] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

where $[T]$ is called the ***transformation matrix***.

As shown in Fig. 1.15, the stress components in the (x, y, z) coordinates are

$$\sigma_x,\ \sigma_y, \sigma_z,\ \tau_{xy},\ \tau_{xz}, \text{ and } \tau_{yz}$$

and the stress components in the (x', y', z') coordinates are

$$\sigma_{x'},\ \sigma_{y'},\sigma_{z'},\ \tau_{x'y'},\ \tau_{x'z'},\ \text{and}\ \tau_{y'z'}$$

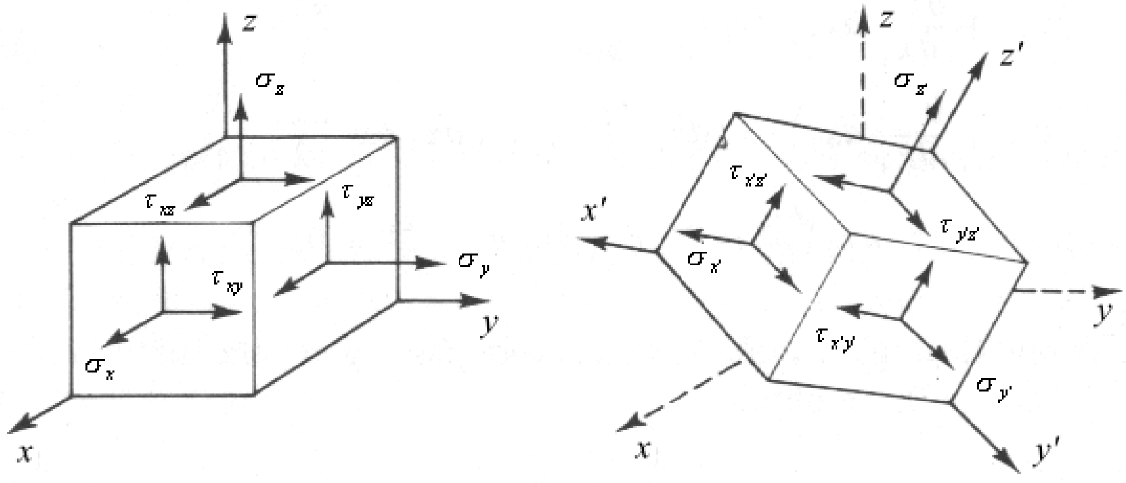


Fig. 1.15

From the previous discussions, the normal stress component $\sigma_{x'}$ is the components in the x' direction of the stress vector on a plane perpendicular to the x'-axis. Thus, from the equation $\sigma_{PN} = \lfloor l \quad m \quad n \rfloor \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$, we have

$$\sigma_{x'} = \lfloor l_1 \quad m_1 \quad n_1 \rfloor \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} l_1 \\ m_1 \\ n_1 \end{Bmatrix}$$

$$\sigma_{x'} = \sigma_x l_1^2 + \sigma_y m_1^2 + \sigma_z n_1^2 + 2m_1 n_1 \tau_{yz} + 2n_1 l_1 \tau_{xz} + 2l_1 m_1 \tau_{xy}$$

Similarly,

$$\sigma_{y'} = \lfloor l_2 \quad m_2 \quad n_2 \rfloor \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} l_2 \\ m_2 \\ n_2 \end{Bmatrix}$$

$$\sigma_{y'} = \sigma_x l_2^2 + \sigma_y m_2^2 + \sigma_z n_2^2 + 2m_2 n_2 \tau_{yz} + 2n_2 l_2 \tau_{xz} + 2l_2 m_2 \tau_{xy}$$

$$\sigma_{z'} = \lfloor l_3 \quad m_3 \quad n_3 \rfloor \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} l_3 \\ m_3 \\ n_3 \end{Bmatrix}$$

$$\sigma_{z'} = \sigma_x l_3^2 + \sigma_y m_3^2 + \sigma_z n_3^2 + 2m_3 n_3 \tau_{yz} + 2n_3 l_3 \tau_{xz} + 2l_3 m_3 \tau_{xy}$$

The shear stress component $\tau_{x'y'}$ is the components in the y' direction of the stress vector on a plane perpendicular to the x'-axis. Thus,

$$\tau_{x'y'} = \lfloor l_1 \quad m_1 \quad n_1 \rfloor \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} l_2 \\ m_2 \\ n_2 \end{Bmatrix}$$

or

$$\tau_{x'y'} = \lfloor l_2 \quad m_2 \quad n_2 \rfloor \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} l_1 \\ m_1 \\ n_1 \end{Bmatrix}$$

$$\begin{aligned} \tau_{x'y'} &= \sigma_x l_1 l_2 + \sigma_y m_1 m_2 + \sigma_z n_1 n_2 + \\ &\quad (m_1 n_2 + m_2 n_1)\tau_{yz} + (n_1 l_2 + n_2 l_1)\tau_{xz} + (l_1 m_2 + l_2 m_1)\tau_{xy} \end{aligned}$$

Similarly,

$$\tau_{x'z'} = \lfloor l_1 \quad m_1 \quad n_1 \rfloor \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} l_3 \\ m_3 \\ n_3 \end{Bmatrix}$$

$$\begin{aligned} \tau_{x'z'} &= \sigma_x l_1 l_3 + \sigma_y m_1 m_3 + \sigma_z n_1 n_3 + \\ &\quad (m_1 n_3 + m_3 n_1)\tau_{yz} + (n_1 l_3 + n_3 l_1)\tau_{xz} + (l_1 m_3 + l_3 m_1)\tau_{xy} \end{aligned}$$

$$\tau_{y'z'} = \lfloor l_2 \quad m_2 \quad n_2 \rfloor \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} l_3 \\ m_3 \\ n_3 \end{Bmatrix}$$

$$\begin{aligned} \tau_{y'z'} &= \sigma_x l_2 l_3 + \sigma_y m_2 m_3 + \sigma_z n_2 n_3 + \\ &\quad (m_2 n_3 + m_3 n_2)\tau_{yz} + (n_2 l_3 + n_3 l_2)\tau_{xz} + (l_2 m_3 + l_3 m_2)\tau_{xy} \end{aligned}$$

By analogous to $\sigma_{PN} = \lfloor N \rfloor [\sigma] \lfloor N \rfloor^T$, we can write down the stress vector in the (x', y', z') coordinates in the matrix form as

$$\begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{x'y'} & \sigma_{y'} & \tau_{y'z'} \\ \tau_{x'z'} & \tau_{y'z'} & \sigma_{z'} \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{x'y'} & \sigma_{y'} & \tau_{y'z'} \\ \tau_{x'z'} & \tau_{y'z'} & \sigma_{z'} \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}^T$$

or, in short

$$[\sigma'] = [T][\sigma][T]^T$$

Principal Normal Stresses

Three Dimensions

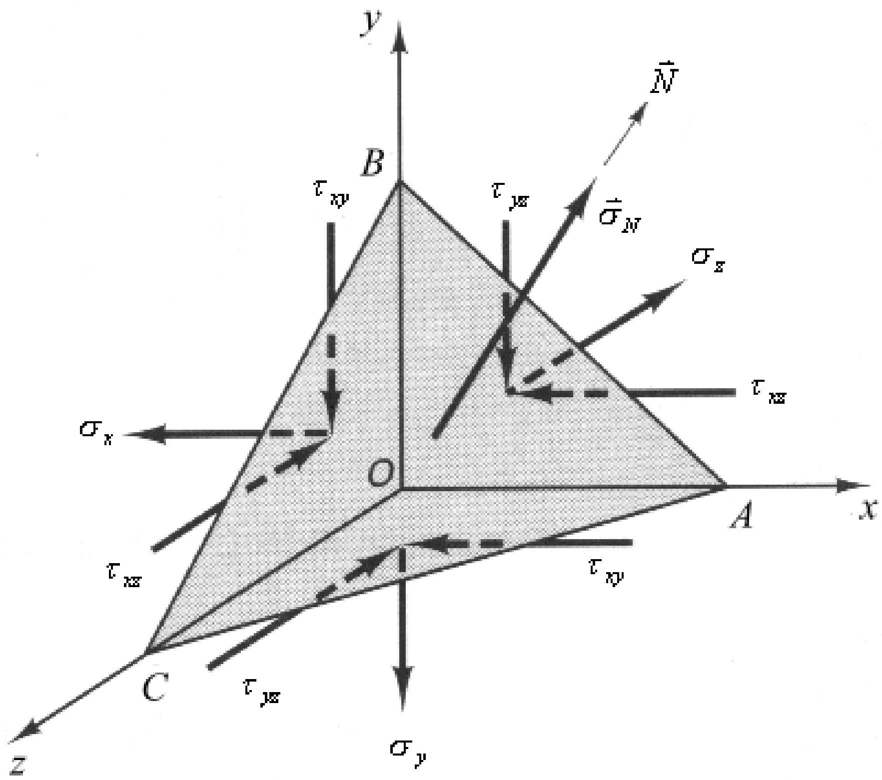


Fig. 1.16

From section 1-3, since the shearing stresses vanish on the principal plane ABC, the principal stress vector $\vec{\sigma}_N$ is the only stress vector acting on the principal plane and in the direction of the unit normal vector $\vec{N}$ to the principal plane.

$$\vec{N} = l\hat{i} + m\hat{j} + n\hat{k}$$

If the infinitesimal area ABC as shown in Fig. 1.16 is defines as dA_{ABC}, then,

$$dA_{0BC} = l dA_{ABC}$$

$$dA_{0AC} = m dA_{ABC}$$

$$dA_{0AB} = n dA_{ABC}$$

The projection of the principal stress vector $\vec{\sigma}_N$ along the coordinate (x, y, z) are

$$\vec{\sigma}_N = \sigma_N l\hat{i} + \sigma_N m\hat{j} + \sigma_N n\hat{k}$$

From the force equilibrium equations, we have

$\sum F_x = 0;$ $$(\sigma_N \, dA_{ABC})l - \sigma_x \, dA_{ABC} l - \tau_{xy} \, dA_{ABC} m - \tau_{xz} \, dA_{ABC} n = 0$$

$\sum F_y = 0;$ $$(\sigma_N \, dA_{ABC})m - \sigma_y \, dA_{ABC} m - \tau_{yz} \, dA_{ABC} n - \tau_{xy} \, dA_{ABC} l = 0$$

$\sum F_z = 0;$ $$(\sigma_N \, dA_{ABC})n - \sigma_z \, dA_{ABC} n - \tau_{xz} \, dA_{ABC} l - \tau_{yz} \, dA_{ABC} m = 0$$

$$(\sigma_x - \sigma_N)l + \tau_{xy}m + \tau_{xz}n = 0$$

$$\tau_{xy}l + (\sigma_y - \sigma_N)m + \tau_{yz}n = 0$$

$$\tau_{xz}l + \tau_{yz}m + (\sigma_z - \sigma_N)n = 0$$

These 3 equations are linear homogeneous equations. Since all three direction cosines can not be zero ($l^2 + m^2 + n^2 = 1$), the system of the linear homogeneous equations has a nontrivial solution if and only if the determinant of the coefficients of l, m, and n vanish. Thus, we have

$$\begin{vmatrix} \sigma_x - \sigma_N & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_N & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_N \end{vmatrix} = 0$$

Expansion of the determinant gives

$$\sigma_N^3 - I_1\sigma_N^2 + I_2\sigma_N - I_3 = 0$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2)$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix} = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - (\sigma_x\tau_{yz}^2 + \sigma_y\tau_{zx}^2 + \sigma_z\tau_{xy}^2)$$

This cubic polynomial equation has three roots σ_1, σ_2, and σ_3 which are the principal normal stresses at point 0.

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & \\ 0 & \sigma_2 & 0 \\ 0. & 0 & \sigma_3 \end{bmatrix} \qquad \sigma_1 \geq \sigma_2 \geq \sigma_3$$

The magnitudes and directions of the principal stresses σ_1, σ_2, and σ_3 for a given member depend only on the loads being applied to the member. They are independent upon the choice of initial coordinate system (x, y, z) used to specify the state of stress at point 0. Thus, the constants I_1, I_2, and I_3 must remain the same magnitudes for all the choices of initial coordinate system (x, y, z), and hence they are *invariant of stress*.

Determine the direction cosines of the principal normal stresses σ_1, σ_2, *and* σ_3

The direction cosines of the principal normal stresses $\sigma_i (i = 1,2,3)$ can be determined by:

1. Substituting any one of the three roots $\sigma_i (i = 1,2,3)$ into any two of the cubic equation, we obtain

$$(\sigma_x - \sigma_i) l_i + \tau_{xy} m_i + \tau_{xz} n_i = 0$$

$$\tau_{xy} l_i + (\sigma_y - \sigma_i) m_i + \tau_{yz} n_i = 0$$

$$\tau_{xz} l_i + \tau_{yz} m_i + (\sigma_z - \sigma_i) n_i = 0$$

2. Solving two of the above three equations together with the equation $l_i^2 + m_i^2 + n_i^2 = 1$ for the direction cosines l_i, m_i, and n_i.

For example, if we need to find the direction cosines of the principal stress σ_1 or l_1, m_1, and n_1.

- Substituting the principal stress σ_1 into any two of the cubic equation such as

$$(\sigma_x - \sigma_1) l_1 + \tau_{xy} m_1 + \tau_{xz} n_1 = 0$$

$$\tau_{xy} l_1 + (\sigma_y - \sigma_1) m_1 + \tau_{yz} n_1 = 0$$

- Dividing the above equations by n_1

$$(\sigma_x - \sigma_1) \frac{l_1}{n_1} + \tau_{xy} \frac{m_1}{n_1} + \tau_{xz} = 0$$

$$\tau_{xy} \frac{l_1}{n_1} + (\sigma_y - \sigma_1) \frac{m_1}{n_1} + \tau_{yz} = 0$$

- Solving the equations for $\frac{l_1}{n_1}$ and $\frac{m_1}{n_1}$
- Substituting the direction cosines l_1 and m_1 which are the functions of the direction cosine n_1 into the equation $l_1^2 + m_1^2 + n_1^2 = 1$, and solving for n_1

Two Dimensions

Consider the plate structure subjected only to the external load parallel to the plate as shown in Fig. 1.17. If the plate is very thin compared to the dimension of the plate, the stresses σ_z, τ_{xz}, and τ_{yx} on an infinitesimal small element far away from the loading points are approximately equal to zero. In addition, let us assume that the remaining stress components σ_x, σ_y, and τ_{xy} are independent of z. This kind of state of stresses on the infinitesimal small element is called the *plane stress* as shown in Fig. 1.18.

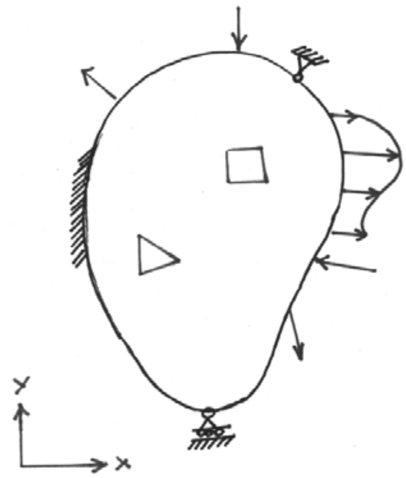


Fig. 1.17

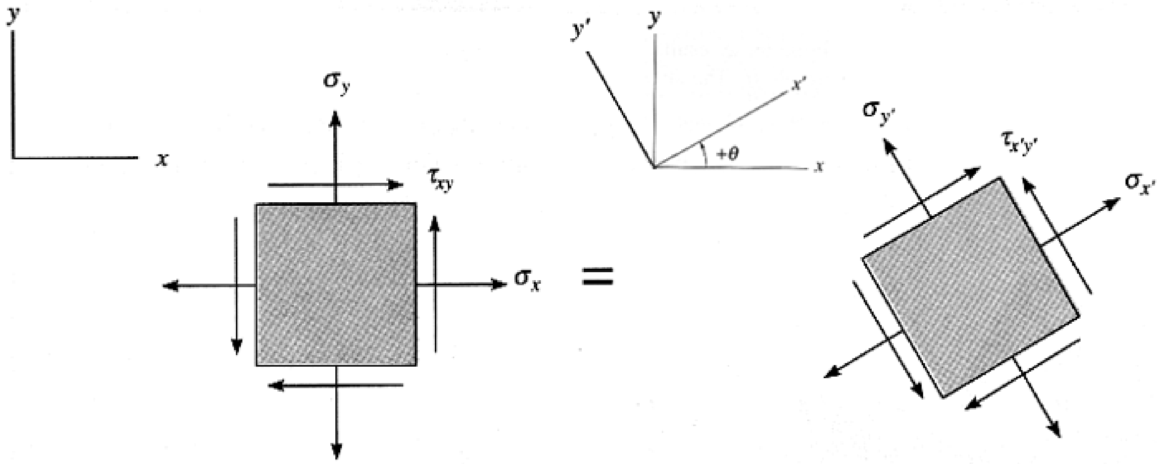


Fig. 1.18

In the similar fashion as for the coordinate transformation in three dimensions, we can determine the coordinate transformation matrix of the plane stress as following.

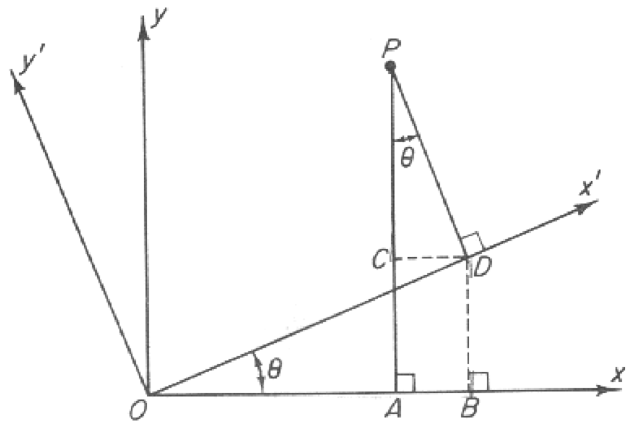


Fig. 1.19

Let (x , y) and (x' , y') denote two rectangular coordinate systems with a common origin as shown in Fig. 1.19. Also, let a general point in space P has coordinate (x , y) and (x' , y') in the respective coordinate system. The angle between the coordinate axes (x , y)

and (x', y') is θ. Therefore, the relationship between the has coordinate (x, y) and (x', y') can be written as

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

$$z = z'$$

Thus, in matrix notation, we have

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

The transformation of the stress components from the (x, y) coordinates to the (x', y') coordinates,

$$\begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{x'y'} & \sigma_{y'} & \tau_{y'z'} \\ \tau_{x'z'} & \tau_{y'z'} & \sigma_{z'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \sigma_{z'} \\ \tau_{y'z'} \\ \tau_{x'z'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

where $m = \cos\theta$ and $n = \sin\theta$.

Since $\sigma_z = \tau_{xz} = \tau_{yz} = 0$ in state of plane stresses, then, the stress components in the (x', y') coordinates are

$$\begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

or

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

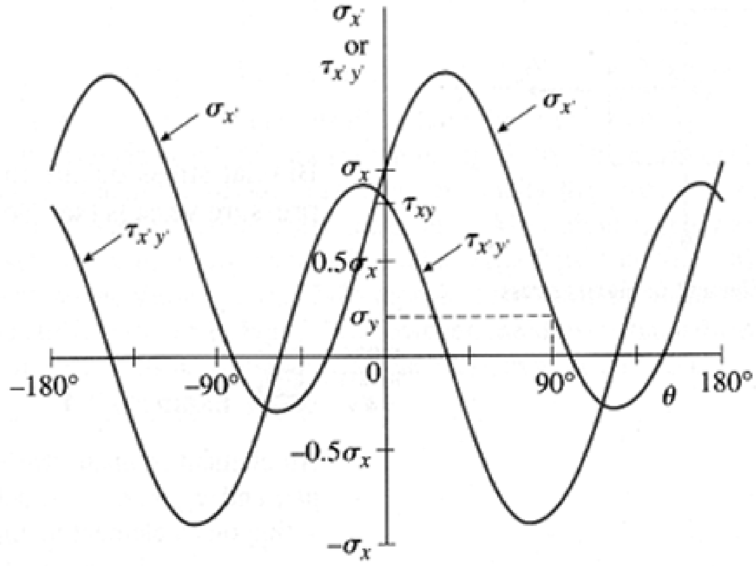


Fig. 1.20

Fig. 1.20 shows the variation of the stress $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ versus θ for $\sigma_y = 0.2\sigma_x$ and $\tau_{xy} = 0.8\sigma_x$.

Mohr's Circle in Two Dimensions

Rewriting the equation of σ'_x and $\tau_{x'y'}$ in the form of

$$\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right) = \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

Squaring each equation and adding the equations together, we have

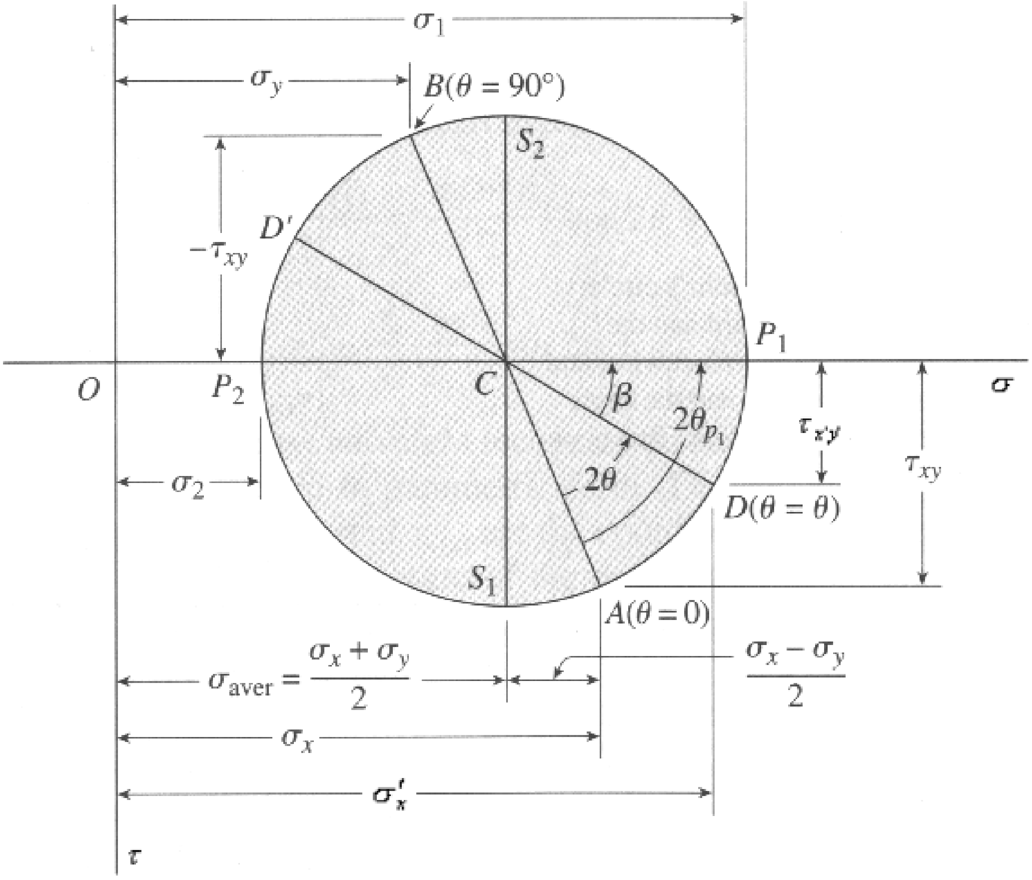
$$\left[\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right]^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

This equation is the equation of a circle in the σ'_x and $\tau_{x'y'}$ plane as shown in Fig. 1.21. The center C of the circle has coordinate

$$(\frac{\sigma_x + \sigma_y}{2}, 0)$$

and radius of the circle is

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



σ_1
σ_y
$B(\theta = 90°)$
S_2
$-\tau_{xy}$
D'
P_1
O
P_2
C
β
$2\theta_{p_1}$
$\tau_{x'y}$
τ_{xy}
σ
σ_2
2θ
$D(\theta = \theta)$
S_1
$A(\theta = 0)$
$\sigma_{aver} = \frac{\sigma_x + \sigma_y}{2}$
$\frac{\sigma_x - \sigma_y}{2}$
σ_x
σ'_x
τ

Fig. 1.21

Principal Shearing Stresses

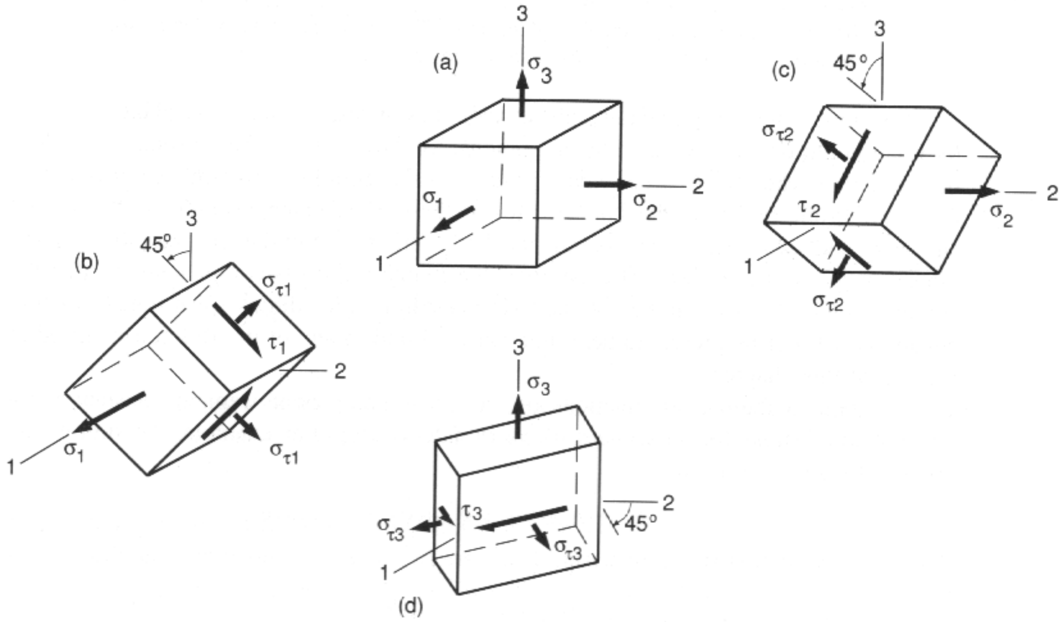


Fig. 1.22

Consider the stress element subjected to the state of principal stresses σ_1, σ_2, and σ_3 as shown in Fig. 1.22a. Also, consider the plane containing any two principal normal stresses, such as σ_2 and σ_3. The maximum shear stress in this plane (the 2-3 plane) occurs on the stress element that is rotated 45^o about the remaining σ_1 principal stress axis as shown in Fig. 1.22b. Since the principal stress σ_1 is independent from the state of stresses on the 2-3 plane, thus, the shear stress $\tau_{\substack{\max \\ 2-3}}$ is one of three principal shear stresses on the stress element and it has the absolute value of

$$\tau_{\substack{\max \\ 2-3}} = \left| \frac{\sigma_2 - \sigma_3}{2} \right|$$

Similarly, the shear stresses $\tau_{\substack{\max \\ 1-3}}$ and $\tau_{\substack{\max \\ 1-2}}$ are the principal shear stresses on the 1-3 plane and the plane 1-2 of the stress element as shown in Fig. 1.22c and 1.22d, respectively. They have the absolute value of

$$\tau_{\substack{\max \\ 1-3}} = \left| \frac{\sigma_1 - \sigma_3}{2} \right|$$

$$\tau_{\substack{\max \\ 1-2}} = \left| \frac{\sigma_1 - \sigma_2}{2} \right|$$

One of the $\tau_{\substack{\max \\ 2-3}}$, $\tau_{\substack{\max \\ 1-3}}$, and $\tau_{\substack{\max \\ 1-2}}$ is the maximum shear stress that occurs for all possible choices of coordinate system of the stress element.

Each plane of the principal shear stress is also acted upon by a normal stress that is the same in the two orthogonal directions.

$\tau_{\underset{2-3}{\max}}$ plane; $$\sigma_{\tau,2-3} = \frac{\sigma_2 + \sigma_3}{2}$$

$\tau_{\underset{1-3}{\max}}$ plane; $$\sigma_{\tau,1-3} = \frac{\sigma_1 + \sigma_3}{2}$$

$\tau_{\underset{1-2}{\max}}$ plane; $$\sigma_{\tau,1-2} = \frac{\sigma_1 + \sigma_2}{2}$$

Mohr's Circles for the Principal Planes

Consider a state of stress that has two components of shear stress equal to zero such as $\tau_{xz} = \tau_{yz} = 0$ as shown in Fig. 1.23.

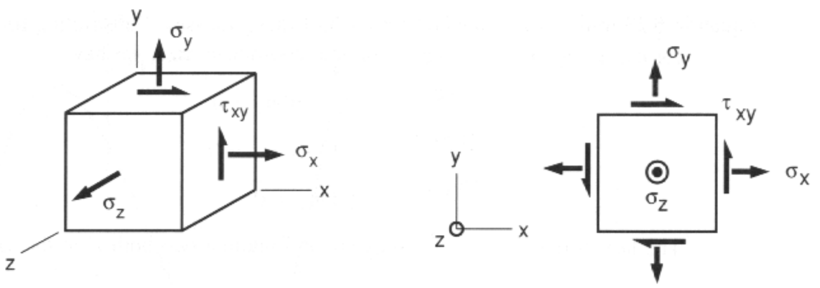


Fig. 1.23

Then, the normal stress σ_z in the direction normal to the plane of the nonzero component of shear stress τ_{xy} is one of the principal stresses of the state of stresses shown.

$$\sigma_z = \sigma_3$$

From the discussions of the plane stress transformation, we can see that the normal stress σ_z does not influence the transformation equations. Thus, in terms of the principal stresses (no shear stress), the Morh's circle can be constructed by using any two of the principal stresses as shown in Fig. 1.24.

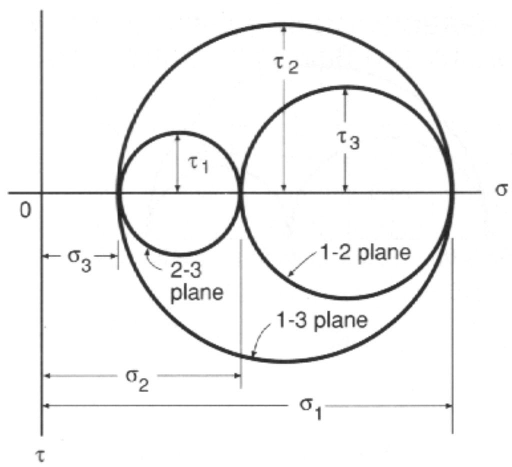


Fig. 1.24

Octahedral Planes and Octahedral Stress

The octahedral planes are the oblique planes that intersect the principal axes $(1,2,3)$ at equal distance from the origin 0 as shown in Fig. 1.25. The unit normal vectors to these planes satisfy the relation

$$l^2 = m^2 = n^2 = \frac{1}{3}$$

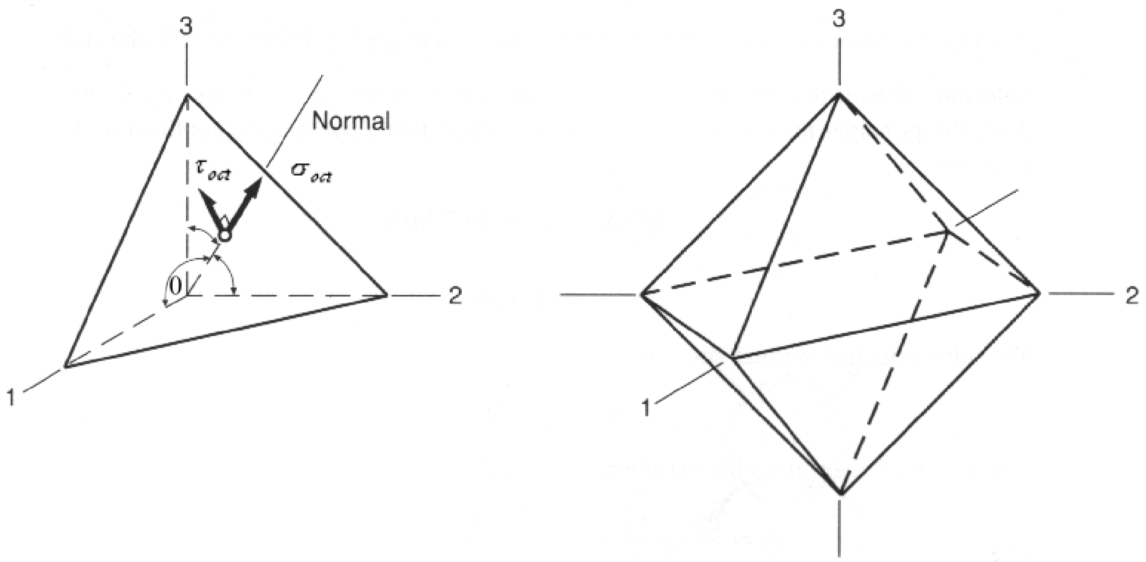


Fig. 1.25

Thus, the octahedral normal stress, σ_{oct} (or the hydrostatic stress) can be determined from the equation

$$\sigma_{PN} = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2mn\tau_{yz} + 2nl\tau_{xz} + 2lm\tau_{xy}$$

Since for the principal axes $\sigma_x = \sigma_1$, $\sigma_y = \sigma_2$, $\sigma_z = \sigma_3$, and $\tau_{xy} = \tau_{xz} = \tau_{yz} = 0$, then

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

In addition, since $\sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_y + \sigma_z$ (see Example 1-1), we have

$$\sigma_{oct} = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{I_1}{3}$$

The the octahedral shearing stress, τ_{oct} can be determined from the equation

$$\sigma_{PS} = \sqrt{\sigma_P^2 - \sigma_{PN}^2} = \sqrt{\sigma_{Px}^2 + \sigma_{Py}^2 + \sigma_{Pz}^2 - \sigma_{PN}^2}$$

$$\tau_{oct} = \sqrt{\frac{1}{3}\sigma_1^2 + \frac{1}{3}\sigma_2^2 + \frac{1}{3}\sigma_3^2 - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2}$$

$$9\tau_{oct}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 = 2I_1^2 - 6I_2$$

Substitution of the stress invariant into τ_{oct}, we have

$$9\tau_{oct}^2 = 2I_1^2 - 6I_2$$
$$= (\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 + 6\tau_{xy}^2 + 6\tau_{xz}^2 + 6\tau_{yz}^2$$

The term σ_{oct} and τ_{oct} are the important quantities as they are used to predict the failure of materials under complex states of stress.

Mean and Deviator Stress

In the theory of plasticity and experiments, it has been shown that yielding and plastic deformations of many metals are independent of the applied normal stress σ_m,

$$\sigma_m = \sigma_{oct} = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{I_1}{3}$$

Therefore, the plastic behavior of the materials is related mainly to the part of stresses that is independent of σ_m.

Rewriting the stress tensor, we have

$$T = T_m + T_d$$

where the stress array T_m is called the *mean* stress tensor or hydrostatic stress tensor, and the stress array T_d is called the *deviator* stress tensor.

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \dfrac{\sigma_x + \sigma_y + \sigma_z}{3} & 0 & 0 \\ 0 & \dfrac{\sigma_x + \sigma_y + \sigma_z}{3} & 0 \\ 0 & 0 & \dfrac{\sigma_x + \sigma_y + \sigma_z}{3} \end{bmatrix} + T_d$$

$$T_d = \begin{bmatrix} \sigma_x - \dfrac{\sigma_x + \sigma_y + \sigma_z}{3} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \dfrac{\sigma_x + \sigma_y + \sigma_z}{3} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \dfrac{\sigma_y + \sigma_y + \sigma_z}{3} \end{bmatrix}$$

$$T_d = \begin{bmatrix} \dfrac{2\sigma_x - \sigma_y - \sigma_z}{3} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \dfrac{2\sigma_y - \sigma_x - \sigma_z}{3} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \dfrac{2\sigma_z - \sigma_y - \sigma_x}{3} \end{bmatrix}$$

If the stress tensor is the principal stress tensor, we can determine the principal values of the stress deviator as following

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \frac{\sigma_1+\sigma_2+\sigma_3}{3} & 0 & 0 \\ 0 & \frac{\sigma_1+\sigma_2+\sigma_3}{3} & 0 \\ 0 & 0 & \frac{\sigma_1+\sigma_2+\sigma_3}{3} \end{bmatrix} + T_d$$

$$T_d = \begin{bmatrix} \sigma_1 - \frac{\sigma_1+\sigma_2+\sigma_3}{3} & 0 & 0 \\ 0 & \sigma_2 - \frac{\sigma_1+\sigma_2+\sigma_3}{3} & 0 \\ 0 & 0 & \sigma_3 - \frac{\sigma_1+\sigma_2+\sigma_3}{3} \end{bmatrix}$$

$$T_d = \begin{bmatrix} \frac{2\sigma_1-\sigma_2-\sigma_3}{3} & 0 & 0 \\ 0 & \frac{2\sigma_2-\sigma_1-\sigma_3}{3} & 0 \\ 0 & 0 & \frac{2\sigma_3-\sigma_1-\sigma_2}{3} \end{bmatrix}$$

$$T_d = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}$$

We can see that $S_1 + S_2 + S_3 = 0$, thus only two of the principal stresses of T_d are independent (we can find the 3rd term from these two).

Example 1-1

Given a state of stresses at a point with respect to a convenient coordinate system (x, y, z) be $\sigma_x = 100$ MPa, $\sigma_y = -60$ MPa, $\sigma_z = 40$ MPa, $\tau_{xy} = 80$ MPa, $\tau_{yz} = \tau_{xz} = 0$ MPa.

a.) Determine the principal normal stresses and the direction cosine of the principal normal stresses

b.) Determine the principal shear stresses.

c.) Determine the octahedral normal stresses and octahedral shearing stresses.

d.) Determine the mean and deviator stress.

The principal normal stresses

$$\begin{vmatrix} 100-\sigma & 80 & 0 \\ 80 & -60-\sigma & 0 \\ 0 & 0 & 40-\sigma \end{vmatrix} = 0$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

where the stress invariants,

$$I_1 = 100 - 60 + 40 = 80 \text{ MPa}$$

$$I_2 = 100(-60) + 100(40) + (-60)40 - 80^2 = -10800 \text{ (MPa)}^2$$

$$I_3 = 100(-60)40 + 2(80)0(0) - 100(0)^2 - (-60)^2(0)^2 - 40(80)^2 = -496000 \text{ (MPa)}^3$$

$$\sigma^3 - 80\sigma^2 - 10800\sigma + 496000 = 0$$

Solving the equation, we obtain

$$\sigma_1 = 133.137 \text{ MPa}$$

$$\sigma_2 = 40 \text{ MPa}$$

$$\sigma_3 = -93.137 \text{ MPa}$$

It should be noted that $\sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_y + \sigma_z = 80$ MPa

The direction cosine of the principal normal stresses

Substituting σ_1 and σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{xz} into the equations

$$(\sigma_x - \sigma_i)l_i + \tau_{xy}m_i + \tau_{xz}n_i = 0 \text{ and } \tau_{xz}l_i + \tau_{yz}m_i + (\sigma_z - \sigma_i)n_i = 0$$

$$(100 - 133.137)l_1 + 80m_1 + 0 = 0$$

$$0 + 0 + (40 - 133.137)n_1 = 0$$

We obtain $n_1 = 0$ and $l_1 = 2.414m_1$. Since $l_1^2 + m_1^2 + n_1^2 = 1$,

$$m_1 = \sqrt{\frac{1}{6.8274}} = 0.383 \text{ and } l_1 = 0.924$$

We can see that the principal axis for σ_1 lie in the $x-y$ plane at the angle of $\alpha = 22.5^o$ counter-clockwise from the $x-$axis.

By using the same calculation procedures, we have the direction cosines of the principal normal stress σ_2 and σ_3 are

$$l_2 = -0.383 \qquad m_2 = 0.924 \qquad n_2 = 0$$

$$l_3 = 0 \qquad m_3 = 0 \qquad n_3 = 1$$

It should be noticed that the principal axis for σ_2 also lie in the $x-y$ plane and is perpendicular to the principal axis for σ_1. In addition, the principal axis for σ_3 is coincident with the original $z-$axis.

The principal shear stresses

$$\tau_{\substack{\max \\ 1-3}} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = \left|\frac{133.137-(-93.137)}{2}\right| = 113.14\,\text{MPa}$$

$$\tau_{\substack{\max \\ 2-3}} = \left|\frac{\sigma_2 - \sigma_3}{2}\right| = \left|\frac{40-(-93.137)}{2}\right| = 66.56\,\text{MPa}$$

$$\tau_{\substack{\max \\ 1-2}} = \left|\frac{\sigma_1 - \sigma_2}{2}\right| = \left|\frac{133.137-40}{2}\right| = 46.56\,\text{MPa}$$

It should be noted that since there is only one nonzero component of shear stress τ_{xy}, the stress normal to the plane of τ_{xy} is one of the principal normal stresses.

$$\sigma_2 = \sigma_z = 40\,\text{MPa}$$

Then, the Mohr's circle as shown in Fig. Ex 1-1 below may be used to determine the principal normal stresses, the direction cosine of the principal normal stresses, and the principal shear stresses in the $x-y$ plane as for the 2-D problem.

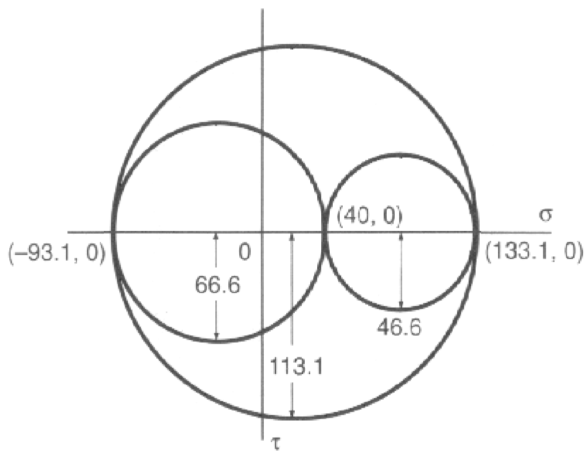


Fig. Ex 1-1

The octahedral normal stresses

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}(133.137 + 40 - 93.137) = 26.667 \text{ MPa}$$

The octahedral shearing stresses

$$\begin{aligned}\tau_{oct} &= \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2} \\ &= \frac{1}{3}\sqrt{(133.137 - 40)^2 + (133.137 - (-93.137))^2 + (40 - (-93.137))^2} \\ &= 92.856 \text{ MPa}\end{aligned}$$

The mean stress

$$T_m = \begin{bmatrix} \dfrac{133.137 + 40 - 93.137}{3} & 0 & 0 \\ 0 & \dfrac{133.137 + 40 - 93.137}{3} & 0 \\ 0 & 0 & \dfrac{133.137 + 40 - 93.137}{3} \end{bmatrix}$$

$$T_m = \begin{bmatrix} 26.667 & 0 & 0 \\ 0 & 26.667 & 0 \\ 0 & 0 & 26.667 \end{bmatrix} \text{MPa}$$

The deviator stress

$$T_d = \begin{bmatrix} \dfrac{2(133.137) - 40 - (-93.137)}{3} & 0 & 0 \\ 0 & \dfrac{2(40) - 133.137 - (-93.137)}{3} & 0 \\ 0 & 0 & \dfrac{2(-93.137) - 133.137 - 40}{3} \end{bmatrix}$$

$$T_d = \begin{bmatrix} 106.470 & 0 & 0 \\ 0 & 13.333 & 0 \\ 0 & 0 & -119.804 \end{bmatrix} \text{MPa}$$

Example 1-2

Given the state of stresses at the dark spot on the surface of a pressure vessel as shown in Fig. Ex 1-2a.

a.) Determine the state of principal stresses.

b.) Determine the state of maximum in-plane shear stress.

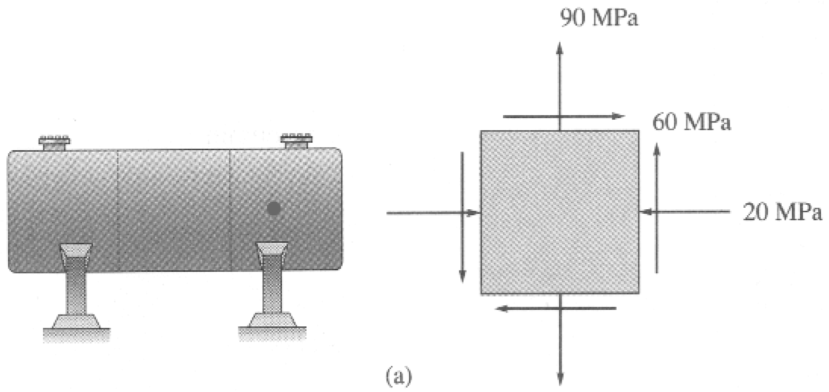


Fig. Ex 1-2a

From the element right-hand face and the sign convention,

$$\sigma_x = -20 \text{ MPa} \qquad \sigma_y = +90 \text{ MPa} \qquad \tau_{xy} = 60 \text{ MPa}$$

The center of the Mohr's circle is located at $(\sigma_{avg}, 0)$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{-20+90}{2} = 35 \text{ MPa}$$

The radius of the Mohr's circle is

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{-20-90}{2}\right)^2 + 60^2} = 81.4 \text{ MPa}$$

Then, the initial point $A\,(-20,+60)$ and the center $C(35,0)$ are plotted on the Mohr's circle as shown in Fig. Ex 1-2b.

Determine the state of principal stresses

The principal stresses are indicated by the coordinate of points B and D on Mohr's circle.

$$\sigma_1 = 35 + 81.4 = 116.4 \text{ MPa}$$

$$\sigma_2 = 35 - 81.4 = -46.4 \text{ MPa}$$

The orientation of the element is determined by calculating the counterclockwise angle $2\theta_{p_1}$ from the radius line AC to BC.

$$2\theta_{p_1} = 180^o - \phi = 180^o - \tan^{-1}\frac{60}{55} = 132.5^o$$

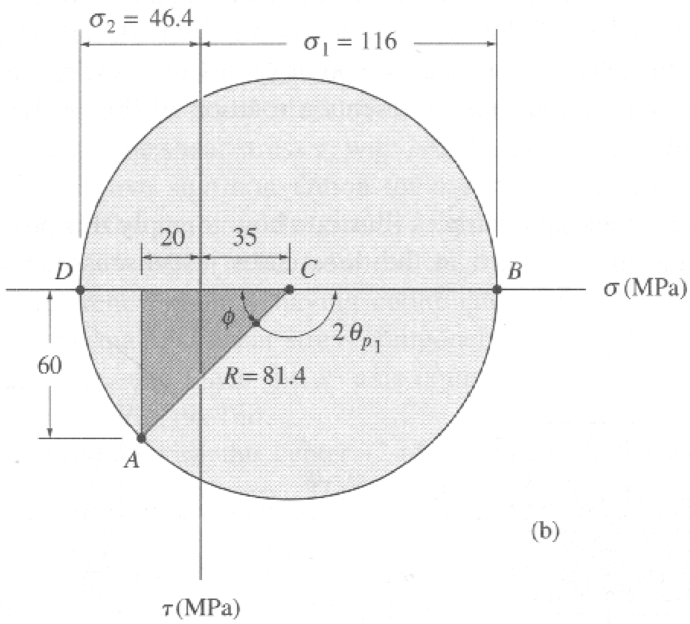


Fig. Ex 1-2b

Thus, the orientation of the planes that contain the state of the principal stresses is

$$\theta_{p1} = \frac{132.5^o}{2} = 66.3^o$$

Fig. Ex 1-2c shows the state of the principal stresses.

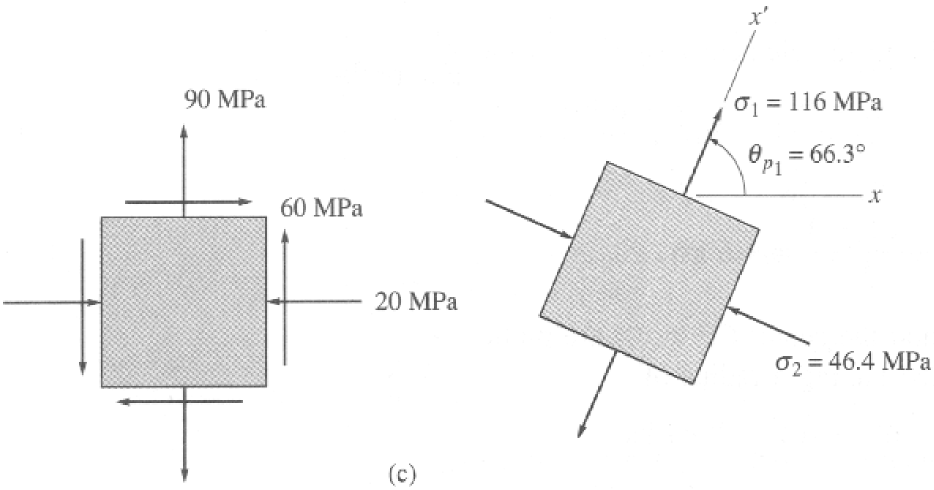


Fig. Ex 1-2c

The state of maximum in-plane shear stress

The maximum in-plane shear stress and the average normal stress are identified by the point E and F on the Mohr's circle in Fig. Ex 1-2d. Hence, we have

$$\tau_{\substack{\max \\ in-plane}} = 81.4 \text{ MPa}$$

and

$$\sigma_{avg} = 35\,\text{MPa}$$

The counterclockwise angle $2\theta_{s1_1}$ from the radius line AC to the radius line EC is

$$2\theta_{s_1} = 90^o - \phi = 90^o - 47.5^o = 42.5^o$$

Thus, the orientation of the planes that contain the state of the maximum in-plane shear stress is

$$\theta_{s1} = \frac{42.5^o}{2} = 21.3^o$$

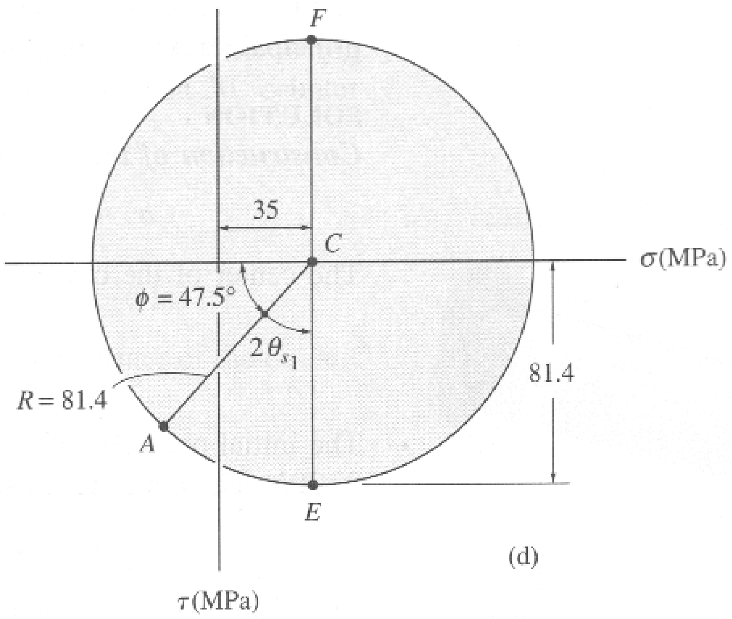


Fig. Ex 1-2d

Fig. Ex 1-2e shows the state of the maximum in-plane shear stress.

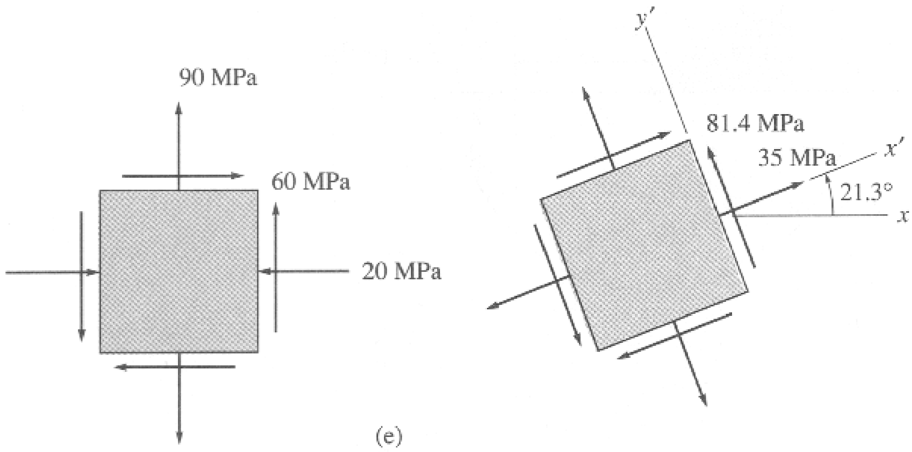


Fig. Ex 1-2e

1.5 Differential Equations of Equilibrium of a Deformable Body

Consider Fig. 1.26, the differential equations of equilibrium in rectangular coordinate axes (x, y, z) can be written as following,

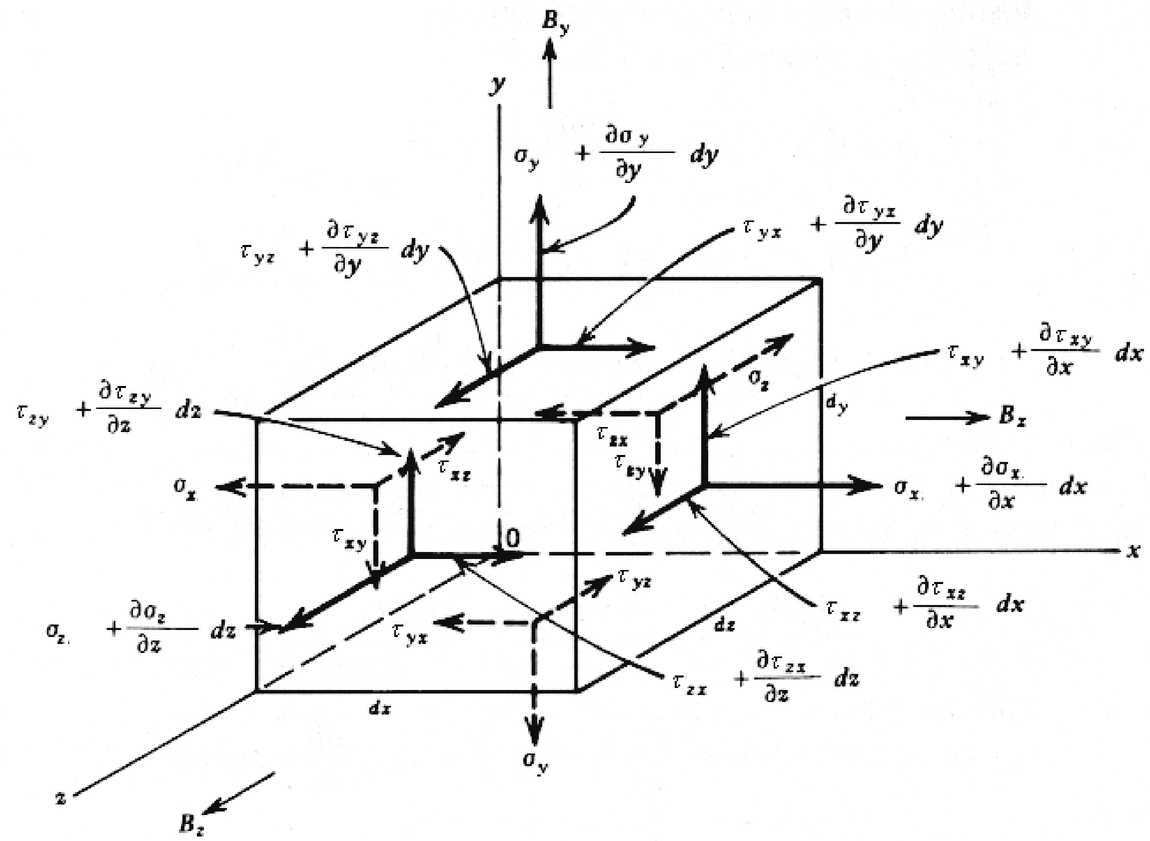


Fig. 1.26

$$\sum F_x = 0; \qquad (\sigma_x + \frac{\partial \sigma_x}{\partial x} dx - \sigma_x)\, dydz + (\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy - \tau_{xy})\, dxdz$$

$$+ (\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} dz - \tau_{xz})\, dxdy + B_x (dxdydz) = 0$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0$$

where B_x is the body force per unit volume in the x direction including the inertia forces. Similarly,

$\sum F_y = 0;$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + B_y = 0$$

$\sum F_z = 0;$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + B_z = 0$$

Equilibrium Equations: Plane Problem

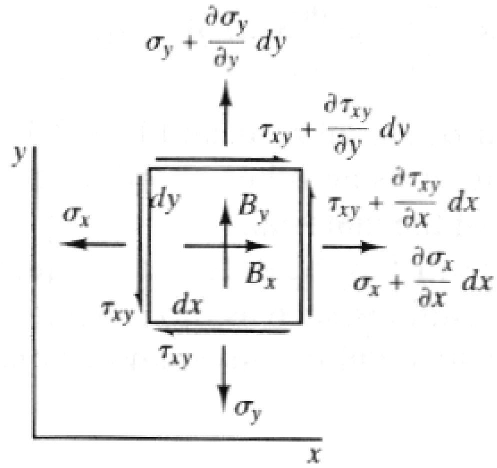


Fig. 1.27

If the body is a plane body of uniform thickness as shown in Fig. 1.27, the differential equations of equilibrium of the three-dimensional body can be reduced to

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + B_y = 0$$

Equilibrium Equations in Polar Coordinate: Plane Problem

Consider a plane body of uniform thickness in polar coordinate (r, θ) as shown in Fig. 1.28, the differential equations of motion of this plane body can be written as following,

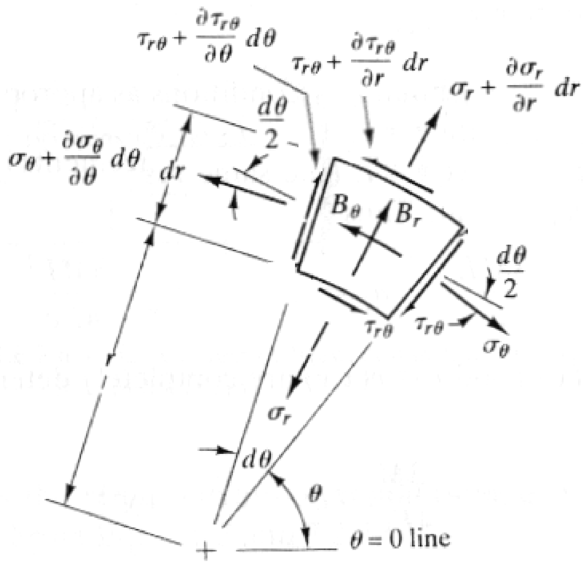


Fig. 1.28

$\sum F_r = 0;$

$$-\sigma_r(rd\theta) - \tau_{r\theta}(dr) - \sigma_\theta \sin\frac{d\theta}{2}(dr) + \left[\sigma_r + \frac{\partial \sigma_r}{\partial r} dr\right](r+dr)d\theta +$$

$$\left[\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} d\theta\right] dr - \left[\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta\right] \sin\frac{d\theta}{2}(dr) + B_r r dr d\theta = 0$$

$\sum F_\theta = 0;$

$$-\tau_{r\theta}(rd\theta) - \sigma_\theta(dr) - \tau_{r\theta}\sin\frac{d\theta}{2}(dr) + \left[\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r} dr\right](r+dr)d\theta +$$

$$\left[\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta\right] dr - \left[\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} d\theta\right] \sin\frac{d\theta}{2}(dr) + B_\theta r dr d\theta = 0$$

Expanding the above two equations, setting $\sin\frac{d\theta}{2} = \frac{d\theta}{2}$, and neglecting the higher-order terms, we have the equilibrium equations in the polar coordinate.

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + B_r = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + 2\frac{\tau_{r\theta}}{r} + B_\theta = 0$$

1.6 Deformation of a Deformable Body

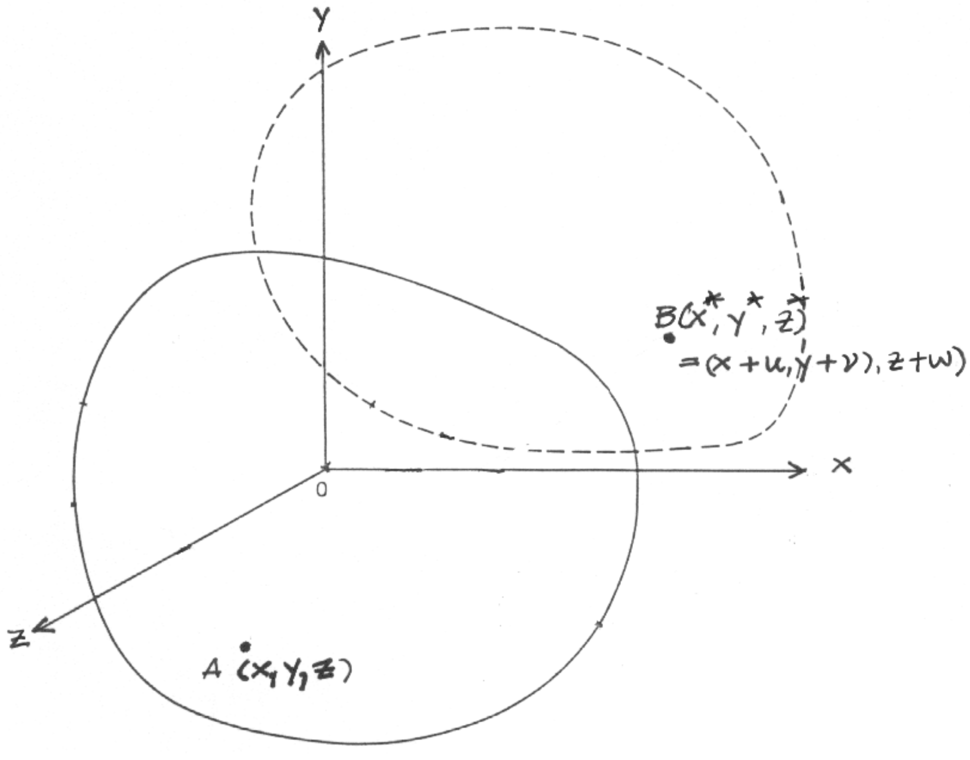


Fig. 1.29

Consider a deformation body in equilibrium as shown in Fig. 1.29 subjected to external loads and deformed to a new equilibrium position indicated by the dashed line. The coordinate of point A on an undeformed is (x, y, z) and, after the deformation, the point A is moved to the point B having the new coordinate of (x^*, y^*, z^*). Noting that

$$x^* = x + u$$

$$y^* = y + v$$

$$z^* = z + w$$

Therefore, in Lagrangian coordinate method, we can write down the relationship between the coordinate x^*, y^*, z^* and x, y, z in the form of

$$x^* = x^*(x, y, z)$$

$$y^* = y^*(x, y, z)$$

$$z^* = z^*(x, y, z)$$

The functions need to be continuous and differentiable with respect to the independent variables. Discontinuity implies rupture of the body.

The total differential equation of the coordinate x^*, y^*, z^* can be written as

$$dx^* = \frac{\partial x^*}{\partial x}dx + \frac{\partial x^*}{\partial y}dy + \frac{\partial x^*}{\partial z}dz$$

$$dy^* = \frac{\partial y^*}{\partial x}dx + \frac{\partial y^*}{\partial y}dy + \frac{\partial y^*}{\partial z}dz$$

$$dz^* = \frac{\partial z^*}{\partial x}dx + \frac{\partial z^*}{\partial y}dy + \frac{\partial z^*}{\partial z}dz$$

1.7 Strain Theory: Principal Strains

Let us define the *engineering strain* ε_E of the line element ds that is transformed to the line element ds^* as

$$\varepsilon_E = \frac{ds^* - ds}{ds}$$

and the quantity M or *magnification factor* as

$$M = \frac{1}{2}\left[\left(\frac{ds^*}{ds}\right)^2 - 1\right] = \frac{1}{2}\left[(1+\varepsilon_E)^2 - 1\right] = \varepsilon_E + \frac{1}{2}\varepsilon_E^2$$

In general, since we have infinite number of particles neighboring to point A, let us consider a particle at the neighboring of point A as defined by the line vector $d\bar{r}_1$ from point A to the particle as shown in Fig. 1.30 and

$$d\bar{r}_1 = dx_1\hat{i} + dy_1\hat{j} + dz_1\hat{k}$$

The magnitude of the line vector $d\bar{r}_1$ is infinitesimal and equal to

$$dr_1 = \sqrt{dx_1^2 + dy_1^2 + dz_1^2}$$

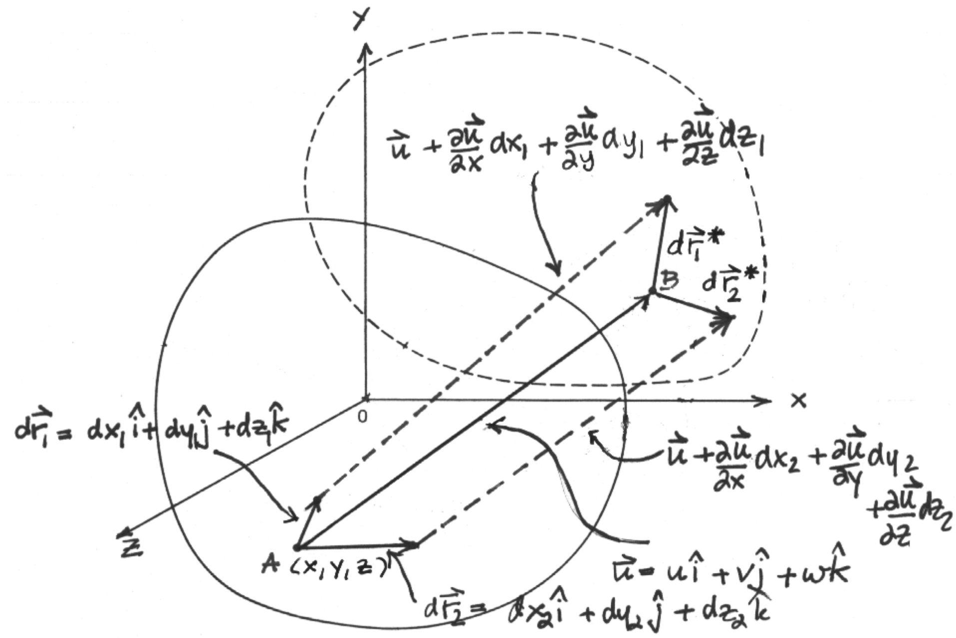


Fig. 1.30

By the deformation, the line vector $d\vec{r}_1$ is moved to be the line vector $d\vec{r}_1^*$ as shown in Fig. 1.30. The vector linked between these two vectors is

$$\vec{u} + \frac{\partial \vec{u}}{\partial x} dx_1 + \frac{\partial \vec{u}}{\partial y} dy_1 + \frac{\partial \vec{u}}{\partial z} dz_1 =$$

$$(u + \frac{\partial u}{\partial x} dx_1 + \frac{\partial u}{\partial y} dy_1 + \frac{\partial u}{\partial z} dz_1)\hat{i} + (v + \frac{\partial v}{\partial x} dx_1 + \frac{\partial v}{\partial y} dy_1 + \frac{\partial v}{\partial z} dz_1)\hat{j}$$

$$+ (w + \frac{\partial w}{\partial x} dx_1 + \frac{\partial w}{\partial y} dy_1 + \frac{\partial w}{\partial z} dz_1)\hat{k}$$

Thus, from Fig. 1.30, we can determine the line vector $d\vec{r}_1^*$ from the relationship

$$\vec{u} + d\vec{r}_1^* = d\vec{r} + (\vec{u} + \frac{\partial \vec{u}}{\partial x} dx_1 + \frac{\partial \vec{u}}{\partial y} dy_1 + \frac{\partial \vec{u}}{\partial z} dz_1)$$

Then,

$$d\vec{r}_1^* = d\vec{r} + (\vec{u} + \frac{\partial \vec{u}}{\partial x} dx_1 + \frac{\partial \vec{u}}{\partial y} dy_1 + \frac{\partial \vec{u}}{\partial z} dz_1) - \vec{u}$$

$$d\vec{r}_1^* = (dx_1 + \frac{\partial u}{\partial x} dx_1 + \frac{\partial u}{\partial y} dy_1 + \frac{\partial u}{\partial z} dz_1)\hat{i} + (dy_1 + \frac{\partial v}{\partial x} dx_1 + \frac{\partial v}{\partial y} dy_1 + \frac{\partial v}{\partial z} dz_1)\hat{j}$$

$$+ (dz_1 + \frac{\partial w}{\partial x} dx_1 + \frac{\partial w}{\partial y} dy_1 + \frac{\partial w}{\partial z} dz_1)\hat{k}$$

Similarly, let consider another particle point having an infinitesimal small distance from point A as defined by the line vector $d\vec{r}_2$ from point A to the particle as shown in Fig. 1.30 and

$$d\vec{r}_2 = dx_2\hat{i} + dy_2\hat{j} + dz_2\hat{k}$$

$$dr_2 = \sqrt{dx_2^2 + dy_2^2 + dz_2^2}$$

By the deformation, the line vector $d\vec{r}_2$ is moved to be the vector $d\vec{r}_2^{\,*}$.

$$d\vec{r}_2^{\,*} = (dx_2 + \frac{\partial u}{\partial x}dx_2 + \frac{\partial u}{\partial y}dy_2 + \frac{\partial u}{\partial z}dz_2)\hat{i} + (dy_2 + \frac{\partial v}{\partial x}dx_2 + \frac{\partial v}{\partial y}dy_2 + \frac{\partial v}{\partial z}dz_2)\hat{j}$$
$$+ (dz_2 + \frac{\partial w}{\partial x}dx_2 + \frac{\partial w}{\partial y}dy_2 + \frac{\partial w}{\partial z}dz_2)\hat{k}$$

Now, let $\frac{\partial u}{\partial x} = u,_x$, $\frac{\partial u}{\partial y} = u,_y$, $\frac{\partial u}{\partial z} = u,_z$, $\frac{\partial v}{\partial x} = v,_x$, $\frac{\partial v}{\partial y} = v,_y$, …, and $\frac{\partial w}{\partial z} = w,_z$. Then, in the form of the magnification factor M, we consider the change in the vector multiplication (dot product) of the line vector $d\vec{r}_1^{\,*}$ and $d\vec{r}_2^{\,*}$, and the line vector $d\vec{r}_1$ and $d\vec{r}_2$ with respect to the line vector $d\vec{r}_1$ times $d\vec{r}_2$.

$$\frac{d\vec{r}_1^{\,*}.d\vec{r}_2^{\,*} - d\vec{r}_1.d\vec{r}_2}{d\vec{r}_1.d\vec{r}_2} = [(dx_1 + u,_x dx_1 + u,_y dy_1 + u,_z dz_1)(dx_2 + u,_x dx_2 + u,_y dy_2 + u,_z dz_2) +$$
$$(dy_1 + v,_x dx_1 + v,_y dy_1 + v,_z dz_1)(dy_2 + v,_x dx_2 + v,_y dy_2 + v,_z dz_2) +$$
$$(dz_1 + w,_x dx_1 + w,_y dy_1 + w,_z dz_1)(dz_2 + w,_x dx_2 + w,_y dy_2 + w,_z dz_2) -$$
$$(dx_1dx_2 + dy_1dy_2 + dz_1dz_2)]/(dx_1dx_2 + dy_1dy_2 + dz_1dz_2)$$

Expanding all terms of the equation, for example,

$$(dx_1 + u,_x dx_1 + u,_y dy_1 + u,_z dz_1)(dx_2 + u,_x dx_2 + u,_y dy_2 + u,_z dz_2) =$$
$$\underline{dx_1dx_2} + \underline{u,_x dx_1dx_2} + u,_y dx_1dy_2 + u,_z dx_1dz_2 +$$
$$\underline{u,_x dx_1dx_2} + \underline{u,_x^2 dx_1dx_2} + u,_x u,_y dx_1dy_2 + u,_x u,_z dx_1dz_2 +$$
$$u,_y dy_1dx_2 + u,_x u,_y dy_1dx_2 + u,_y^2 dy_1dy_2 + u,_y u,_z dy_1dz_2 +$$
$$u,_z dz_1dx_2 + u,_x u,_z dz_1dx_2 + u,_y u,_z dz_1dy_2 + u,_z^2 dz_1dz_2$$

$$(dy_1 + v,_x dx_1 + v,_y dy_1 + v,_z dz_1)(dy_2 + v,_x dx_2 + v,_y dy_2 + v,_z dz_2) =$$
$$dy_1dy_2 + v,_x dy_1dx_2 + v,_y dy_1dy_2 + v,_z dy_1dz_2 +$$
$$v,_x dx_1dy_2 + \underline{v,_x^2 dx_1dx_2} + v,_x v,_y dx_1dy_2 + v,_x v,_z dx_1dz_2 +$$
$$v,_y dy_1dy_2 + v,_x v,_z dy_1dx_2 + v,_y^2 dy_1dy_2 + v,_y v,_z dy_1dz_2 +$$
$$v,_z dz_1dy_2 + v,_x v,_z dz_1dx_2 + v,_y v,_z dz_1dy_2 + v,_x u,_z dz_1dz_2$$

………

Then, consider only the numerator of the equation, we have

$$d\vec{r}_1^{\,*}.d\vec{r}_2^{\,*} - d\vec{r}_1.d\vec{r}_2 =$$
$$[(u,_x + u,_x + u,_x^2 + v,_x^2 + \ldots)dx_1dx_2 + (u,_y + u,_x u,_y + v,_x + v,_x v,_y + \ldots)dx_1dy_2 +$$

$$(u,_x + u,_x u,_z + v,_x v,_z + \ldots)dx_1 dz_2 +$$

$$(u,_y + u,_x u,_y + v,_x + v,_x v,_z + \ldots)dy_1 dx_2 + (u,_y^2 + v,_y + v,_y + v,_y^2 + \ldots)dy_1 dy_2 +$$

$$(u,_y u,_z + v,_z + v,_y v,_z + \ldots)dy_1 dz_2 +$$

$$(u,_z + u,_x u,_z + v,_x v,_z + \ldots)dz_1 dx_2 + (u,_y u,_z + v,_z + v,_y v,_z + \ldots)dz_1 dy_2 +$$

$$(w,_z + w,_z + w,_z^2 + \ldots)dz_1 dz_2]$$

Rearranging,

$$d\bar{r}_1^* . d\bar{r}_2^* - d\bar{r}_1 . d\bar{r} = 2\varepsilon_x dx_1 dx_2 + \gamma_{xy} dx_1 dy_2 + \gamma_{xz} dx_1 dz_2 +$$

$$\gamma_{yx} dy_1 dx_2 + 2\varepsilon_y dy_1 dy_2 + \gamma_{yz} dy_1 dz_2 +$$

$$\gamma_{zx} dz_1 dx_2 + \gamma_{zy} dz_1 dy_2 + 2\varepsilon_z dz_1 dz_2$$

or in another form, we have

$$\frac{1}{2}\left[d\bar{r}_1^* . d\bar{r}_2^* - d\bar{r}_1 . d\bar{r}_2\right] = \varepsilon_x dx_1 dx_2 + \frac{1}{2}\gamma_{xy} dx_1 dy_2 + \frac{1}{2}\gamma_{xz} dx_1 dz_2 +$$

$$\frac{1}{2}\gamma_{yx} dy_1 dx_2 + \varepsilon_y dy_1 dy_2 + \frac{1}{2}\gamma_{yz} dy_1 dz_2 +$$

$$\frac{1}{2}\gamma_{zx} dz_1 dx_2 + \frac{1}{2}\gamma_{zy} dz_1 dy_2 + \varepsilon_z dz_1 dz_2$$

where

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$\varepsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$\frac{1}{2}\gamma_{xy} = \frac{1}{2}\gamma_{yx} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]$$

$$\frac{1}{2}\gamma_{xz} = \frac{1}{2}\gamma_{zx} = \frac{1}{2}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right]$$

$$\frac{1}{2}\gamma_{yz} = \frac{1}{2}\gamma_{zy} = \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right]$$

which are called the finite strain-displacement relations.

For a special case when the line vector $d\bar{r}_1$ and $d\bar{r}_2$ are identical ($dx_1 = dx_2 = dx$, $dy_1 = dy_2 = dy$, $dz_1 = dz_2 = dz$), we have the dot product of the vectors

$$d\bar{r}_1.d\bar{r}_2 = dx_1dx_2 + dy_1dy_2 + dz_1dz_2 = ds^2$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

By the definition of the direction cosine and let the identical vector $d\bar{r}_1$ and $d\bar{r}_2$ have the direction cosine of $\frac{dx}{ds} = l$, $\frac{dy}{ds} = m$, and $\frac{dz}{ds} = n$, we have

$$\frac{1}{2}\frac{d\bar{r}_1^*.d\bar{r}_2^* - d\bar{r}_1.d\bar{r}_2}{d\bar{r}_1.d\bar{r}_2} = \varepsilon_x\frac{dx^2}{ds^2} + \frac{1}{2}\gamma_{xy}\frac{dxdy}{ds^2} + \frac{1}{2}\gamma_{xz}\frac{dxdz}{ds^2} +$$

$$\frac{1}{2}\gamma_{yx}\frac{dydx}{ds^2} + \varepsilon_y\frac{dy^2}{ds^2} + \frac{1}{2}\gamma_{yz}\frac{dydz}{ds^2} +$$

$$\frac{1}{2}\gamma_{zx}\frac{dzdx}{ds^2} + \frac{1}{2}\gamma_{zy}\frac{dzdy}{ds^2} + \varepsilon_z\frac{dz^2}{ds^2}$$

Thus,

$$M = \frac{1}{2}\left[\frac{(ds^*)^2 - ds^2}{ds^2}\right] = \frac{1}{2}\frac{d\bar{r}_1^*.d\bar{r}_2^* - d\bar{r}_1.d\bar{r}_2}{d\bar{r}_1.d\bar{r}_2} = \varepsilon_x l^2 + \gamma_{xy}lm + \gamma_{xz}nl + \varepsilon_y m^2 + \gamma_{yz}mn + \varepsilon_z n^2$$

or,

$$\varepsilon_E + \frac{1}{2}\varepsilon_E^2 = \varepsilon_x l^2 + \gamma_{xy}lm + \gamma_{xz}nl + \varepsilon_y m^2 + \gamma_{yz}mn + \varepsilon_z n^2$$

Final Direction of Vector $\bar{r}_1$

As a result of the deformation, vector $\bar{r}_1$ deforms into the vector $\bar{r}_1^*$. Let the direction cosines of vector $\bar{r}_1$ and $\bar{r}_1^*$ are

$$\frac{dx}{dr_1} = l,\ \frac{dy}{dr_1} = m,\ \text{and}\ \frac{dz}{dr_1} = n$$

$$\frac{dx^*}{dr_1^*} = l^*,\ \frac{dy^*}{dr_1^*} = m^*,\ \text{and}\ \frac{dz^*}{dr_1^*} = n^*$$

Alternatively, we may write

$$\frac{dx^*}{dr_1}\frac{dr_1}{dr_1^*} = l^*,\ \frac{dy^*}{dr_1}\frac{dr_1}{dr_1^*} = m^*,\ \text{and}\ \frac{dz^*}{dr_1}\frac{dr_1}{dr_1^*} = n^*$$

By using the previously obtained relations, $x^* = x + u$, $y^* = y + v$, $z^* = z + w$, and $dx^* = \frac{\partial x^*}{\partial x}dx + \frac{\partial x^*}{\partial y}dy + \frac{\partial x^*}{\partial z}dz$, $dy^* = \frac{\partial y^*}{\partial x}dx + \frac{\partial y^*}{\partial y}dy + \frac{\partial y^*}{\partial z}dz$, and $dz^* = \frac{\partial z^*}{\partial x}dx + \frac{\partial z^*}{\partial y}dy + \frac{\partial z^*}{\partial z}dz$, we find

$$\frac{dx^*}{dr_1} = \left(1+\frac{\partial u}{\partial x}\right)l + \frac{\partial u}{\partial y}m + \frac{\partial u}{\partial z}n$$

$$\frac{dy^*}{dr_1} = \frac{\partial v}{\partial x}l + \left(1+\frac{\partial y}{\partial y}\right)m + \frac{\partial v}{\partial z}n$$

$$\frac{dz^*}{dr_1} = \frac{\partial w}{\partial x}l + \frac{\partial w}{\partial y}m + \left(1+\frac{\partial w}{\partial z}\right)n$$

By using the engineering strain equation $dr_1 / dr_1^* = 1/(1+\varepsilon_E)$ with the above two equations, we obtain the final direction cosines of vector $\vec{r}_1$ when it passes into the vector $\vec{r}_1^*$ under the deformation in the form of

$$(1+\varepsilon_E)l^* = \left(1+\frac{\partial u}{\partial x}\right)l + \frac{\partial u}{\partial y}m + \frac{\partial u}{\partial z}n$$

$$(1+\varepsilon_E)m^* = \frac{\partial v}{\partial x}l + \left(1+\frac{\partial y}{\partial y}\right)m + \frac{\partial v}{\partial z}n$$

$$(1+\varepsilon_E)n^* = \frac{\partial w}{\partial x}l + \frac{\partial w}{\partial y}m + \left(1+\frac{\partial w}{\partial z}\right)n$$

Definition of Shear Strain

If originally the vectors $d\vec{r}_1$ and $d\vec{r}_2$ having the direction cosine l_1, m_1, n_1 and l_2, m_2, n_2, respectively, are normal to each other, by the definition of scalar product of vectors

$$\cos\frac{\pi}{2} = l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

If the angle between the vectors $d\vec{r}_1^*$ and $d\vec{r}_2^*$ having the direction cosine l_1^*, m_1^*, n_1^* and l_2^*, m_2^*, n_2^*, respectively, is equal to θ^* after the deformation, by the definition of scalar product of vectors

$$\cos\theta^* = l_1^* l_2^* + m_1^* m_2^* + n_1^* n_2^*$$

Then, the angle change can be determined from

$$d\vec{r}_1^* . d\vec{r}_2^* - d\vec{r}_1 . d\vec{r}_2 = dr_1^* dr_2^* \cos\theta^* - 0 = dr_1^* dr_2^* \cos\theta^*$$

By using the definition of the magnification factor where $dr_1^* / dr_1 = 1+\varepsilon_{E1}$ and $dr_1^* / dr_1 = 1+\varepsilon_{E1}$, we have the *engineering shearing strain* γ_{12} in the form of

$$\gamma_{12} = \frac{dr_1^*}{dr_1}\frac{dr_2^*}{dr_2}\cos\theta^* = (1+\varepsilon_{E1})(1+\varepsilon_{E2})\left[l_1^* l_2^* + m_1^* m_2^* + n_1^* n\right]$$

Using the equation of the final direction of vector $\vec{r}_1$ when it passes into the vector $\vec{r}_1^*$ and the finite strain-displacement relation, we have the engineering shearing strain between the vector $\vec{r}_1$ and $\vec{r}_2$ as they are deformed into the vector $\vec{r}_1^*$ and $\vec{r}_2^*$ as shown in Fig. 1.30 in the form of

$$\gamma_{12} = 2\varepsilon_x l_1 l_2 + \gamma_{xy} l_1 m_2 + \gamma_{xz} n_2 l_1 + \gamma_{yx} l_2 m_1 + 2\varepsilon_y m_1 m_2 + \gamma_{yz} m_1 n_2 + \gamma_{zx} n_1 l_2 + \gamma_{zy} m_2 n_1 + 2\varepsilon_z n_1 n_2$$

If the strain ε_{E1} and ε_{E2} are small and the angle change are small,

$$\gamma_{12} \approx \frac{\pi}{2} - \theta^*$$

and the engineering shearing strain becomes approximately equal to the change in angle between the vector $\vec{r}_1$ and $\vec{r}_2$.

Strain transformation

The strain tensor,

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \varepsilon_y & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \varepsilon_z \end{bmatrix}$$

The strain tensor obeys the tensor law of transformation when the coordinates are changed as the stress tensor.

The transformation of the strain components from the (x, y) coordinates to the (x', y', z') coordinates,

$$\begin{bmatrix} \varepsilon_{x'} & \varepsilon_{x'y'} & \varepsilon_{x'z'} \\ \varepsilon_{x'y'} & \varepsilon_{y'} & \varepsilon_{y'z'} \\ \varepsilon_{x'z'} & \varepsilon_{y'z'} & \varepsilon_{z'} \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}^T$$

Performing the matrix operations, we have

$$\varepsilon_{x'} = \varepsilon_x l_1^2 + \varepsilon_y m_1^2 + \varepsilon_z n_1^2 + m_1 n_1 \gamma_{yz} + n_1 l_1 \gamma_{xz} + l_1 m_1 \gamma_{xy}$$

$$\varepsilon_{y'} = \varepsilon_x l_2^2 + \varepsilon_y m_2^2 + \varepsilon_z n_2^2 + m_2 n_2 \gamma_{yz} + n_2 l_2 \gamma_{xz} + l_2 m_2 \gamma_{xy}$$

$$\varepsilon_{z'} = \varepsilon_x l_3^2 + \varepsilon_y m_3^2 + \varepsilon_z n_3^2 + m_3 n_3 \gamma_{yz} + n_3 l_3 \gamma_{xz} + l_3 m_3 \gamma_{xy}$$

$$\begin{aligned} \gamma_{x'y'}/2 = {} & \varepsilon_x l_1 l_2 + \varepsilon_y m_1 m_2 + \varepsilon_z n_1 n_2 + \\ & (m_1 n_2 + m_2 n_1)\gamma_{yz}/2 + (n_1 l_2 + n_2 l_1)\gamma_{xz}/2 + (l_1 m_2 + l_2 m_1)\gamma_{xy}/2 \end{aligned}$$

$$\begin{aligned} \gamma_{x'z'}/2 = {} & \varepsilon_x l_1 l_3 + \varepsilon_y m_1 m_3 + \varepsilon_z n_1 n_3 + \\ & (m_1 n_3 + m_3 n_1)\gamma_{yz}/2 + (n_1 l_3 + n_3 l_1)\gamma_{xz}/2 + (l_1 m_3 + l_3 m_1)\gamma_{xy}/2 \end{aligned}$$

$$\begin{aligned} \gamma_{y'z'}/2 = {} & \varepsilon_x l_2 l_3 + \varepsilon_y m_2 m_3 + \varepsilon_z n_2 n_3 + \\ & (m_2 n_3 + m_3 n_2)\gamma_{yz}/2 + (n_2 l_3 + n_3 l_2)\gamma_{xz}/2 + (l_2 m_3 + l_3 m_2)\gamma_{xy}/2 \end{aligned}$$

Similar to the stress transformation, the transformation of the strain components from the (x, y) coordinates to the (x', y') coordinates in two dimension can be performed as

$$\begin{bmatrix} \varepsilon_{x'} & \varepsilon_{x'y'} & \varepsilon_{x'z'} \\ \varepsilon_{x'y'} & \varepsilon_{y'} & \varepsilon_{y'z'} \\ \varepsilon_{x'z'} & \varepsilon_{y'z'} & \varepsilon_{z'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{x} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{y} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{z} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $\varepsilon_z = \varepsilon_{xz} = \varepsilon_{yx} = 0$ in state of plane strains, and $\gamma_{x'y'} = 2\varepsilon_{x'y'}$ and $\gamma_{xy} = 2\varepsilon_{xy}$, then, the stress components in the (x', y') coordinates is

$$\begin{Bmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \gamma_{x'y'} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{x} \\ \varepsilon_{y} \\ \gamma_{xy} \end{Bmatrix}$$

where $m = \cos\theta$ and $n = \sin\theta$.

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\tau_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

It should be noted that since the equations of plane strain transformation are mathematically similar to the equations of plane stress transformation, Mohr's circle can also be used to solve problems involving the transformation of strain.

Principal strains

Through any point in an undeformed member, there are three mutually perpendicular line elements that remain perpendicular under the deformation. The strains of these line elements are called the *principal strains* at the point. The maximum and minimum strain can be determined by using the 2^{nd} method of calculus of variation.

If we need to determine the maximum and minimum of the function $F = F(x, y, z)$ with the condition $G(x, y, z) = 0$, we assume a function $\overline{F} = F - \lambda G$ and the maximum and minimum values of λ can be determined by solving the simultaneous equations

$$\frac{\partial \overline{F}}{\partial x} = 0, \ \frac{\partial \overline{F}}{\partial y} = 0, \ \frac{\partial \overline{F}}{\partial z} = 0, \ G(x, y, z) = 0$$

for 4 unknowns which are x, y, z, and λ. In this case, we have $F = \varepsilon_x l^2 + \gamma_{xy} lm + \gamma_{xz} nl + \varepsilon_y m^2 + \gamma_{yz} mn + \varepsilon_z n^2$ and $G = l^2 + m^2 + n^2$ -1=0. Then,

$$\overline{F} = \varepsilon_x l^2 + \gamma_{xy} lm + \gamma_{xz} nl + \varepsilon_y m^2 + \gamma_{yz} mn + \varepsilon_z n^2 - \lambda\,(l^2 + m^2 + n^2 - 1)$$

The maximum and minimum values of λ can be determined by solving the simultaneous equations

$$\frac{\partial \overline{F}}{\partial l} = 0; \qquad 2(\varepsilon_x - \lambda)l + \gamma_{xy}m + \gamma_{xz}n = 0$$

$$\frac{\partial \overline{F}}{\partial m} = 0; \qquad \gamma_{xy}l + 2(\varepsilon_y - \lambda)m + \gamma_{yz}n = 0$$

$$\frac{\partial \overline{F}}{\partial n} = 0; \qquad \gamma_{xz}l + \gamma_{yz}m + 2(\varepsilon_z - \lambda)n = 0$$

$$l^2 + m^2 + n^2 = 1$$

For these linear homogenous equations, we may rewrite the first three equations in the determinant form as

$$\begin{vmatrix} (\varepsilon_x - \lambda) & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{xy}/2 & (\varepsilon_y - \lambda) & \gamma_{yz}/2 \\ \gamma_{xz}/2 & \gamma_{yz}/2 & (\varepsilon_z - \lambda) \end{vmatrix} = 0$$

For nontrivial solution, we have

$$\lambda^3 - J_1\lambda^2 + J_2\lambda - J_3 = 0$$

where

$$J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$J_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2)$$

$$J_3 = \begin{vmatrix} \varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{xy}/2 & \varepsilon_y & \gamma_{yz}/2 \\ \gamma_{xz}/2 & \gamma_{yz}/2 & \varepsilon_z \end{vmatrix}$$

Solving for the values of principal strains λ_1, λ_2, and λ_3. Then, substituting λ_i back into the simultaneous equations, we obtain the direction cosine of the principal plane l_i, m_i, and n_i, respectively.

If (x , y , z) are the principal axes, then,

$$J_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$J_2 = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3$$

$$J_3 = \varepsilon_1\varepsilon_2\varepsilon_3$$

In two-dimensions, the principal strains and principal planes can be obtained easily as

$$\varepsilon_{\substack{1\\2}} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

In addition, the maximum in-plane shear strain can be determined from

$$\frac{(\gamma_{x'y'})_{\substack{\text{max}\\ \text{in-plane}}}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\tan 2\theta_s = \frac{-(\varepsilon_x - \varepsilon_y)}{\gamma_{xy}}$$

Example 1-3

Given a state of strains at a point with respect to a convenient coordinate system (x , y , z) be $\varepsilon_x = -3000\ \mu\varepsilon$, $\varepsilon_y = 2000\ \mu\varepsilon$, $\varepsilon_z = -2000\ \mu\varepsilon$, $\gamma_{xy} = -5830\ \mu\varepsilon$, $\gamma_{yz} = -670\ \mu\varepsilon$, and $\gamma_{xz} = -3000\ \mu\varepsilon$.

a.) Determine the principal normal strains and the direction cosine of the principal normal strains

b.) Determine the principal shear strains.

The principal normal strains

The strain invariants,

$$J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$J_1 = (-3000 + 2000 - 2000)10^{-6} = -3(10^{-3})$$

$$J_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2)$$

$$J_2 = \left[(-3000)2000 + 2000(-2000) + (-3000)(-2000) - \frac{1}{4}\left[5830^2 + (-670)^2 + (-3000)^2\right]\right]10^{-12}$$

$$= -14.8595(10^{-6})$$

$$J_3 = \begin{vmatrix} \varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{xy}/2 & \varepsilon_y & \gamma_{yz}/2 \\ \gamma_{xz}/2 & \gamma_{yz}/2 & \varepsilon_z \end{vmatrix}$$

$$J_3 = 2.77607(10^{-8})$$

For nontrivial solutions, we have

$$\lambda^3 + 3(10^{-3}))\lambda^2 - 14.8595(10^{-6})\lambda - 2.77607(10^{-8}) = 0$$

Thus, the principal strains are

$$\varepsilon_1 = \lambda_1 = 0.00350$$

$$\varepsilon_2 = \lambda_2 = -0.00162$$

$$\varepsilon_3 = \lambda_3 = -0.00488$$

The direction cosine of the principal normal strains

Substituting $\lambda_1 = 0.00350$ and ε_x , ε_y , ε_z , γ_{xy} , γ_{yz} , and γ_{xz} into the equations $2(\varepsilon_x - \lambda)l + \gamma_{xy}m + \gamma_{xz}n = 0$ and $\gamma_{xy}l + 2(\varepsilon_y - \lambda)m + \gamma_{yz}n = 0$. After rearranging the equations, we have

$$-0.130\frac{l_1}{n_1} + 0.00583\frac{m_1}{n_1} - 0.003 = 0$$

$$0.00583\frac{l_1}{n_1} + -0.003\frac{m_1}{n_1} - 0.00067 = 0$$

Solving the simultaneous equations, we obtain

$$\frac{l_1}{n_1} = -2.576 \text{ and } \frac{m_1}{n_1} = -5.228$$

Since $l_1^2 + m_1^2 + n_1^2 = 1$, then,

$$n_1 = 0.1691$$

and $m_1 = -0.884$ and $l_1 = -0.4356$

By using the same calculation procedures, we have the direction cosines of the principal normal stress λ_2 and λ_3 are

$$l_2 = -0.328 \quad m_2 = 0.355 \quad n_2 = 0.991$$

$$l_3 = 0.849 \quad m_3 = -0.341 \quad n_3 = 0.403$$

The principal shear strains

By using the same principal used to find the principal shear stresses, the principal shear strains can be determine as following.

$$\frac{\gamma_1}{2} = \left|\frac{\lambda_1 - \lambda_3}{2}\right|$$

$$\gamma_1 = 0.00838$$

$$\frac{\gamma_2}{2} = \left|\frac{\lambda_2 - \lambda_3}{2}\right|$$

$$\gamma_2 = 0.00512$$

$$\frac{\gamma_3}{2} = \left|\frac{\lambda_1 - \lambda_2}{2}\right|$$

$$\gamma_3 = 0.00326$$

1.8 Strain Rosettes

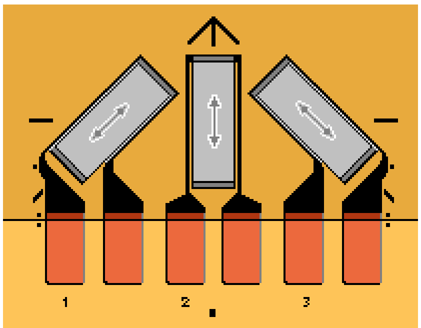


Fig. 1.31

The normal strain at a point of a general testing specimen are usually obtained by using a cluster of three electrical-resistance strain gauges, arranged in a specified pattern called strain rosette as shown in Fig. 1.31 and Fig. 1.32.

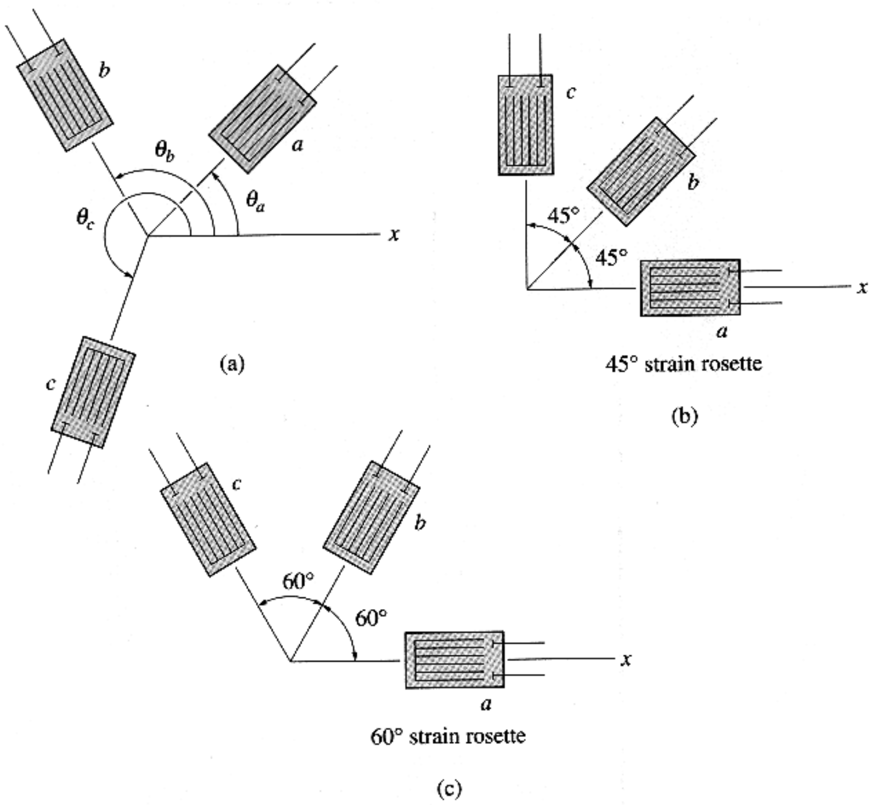


Fig. 1.32

In general, if we know the angles θ_a, θ_b, and θ_c of the strain rosette with respect to an axis as shown in Fig. 1.32a and the measured strains ε_a, ε_b, and ε_c, we can determine the strain ε_x, ε_y, and γ_{xy} from the strain-transformation.

$$\varepsilon_a = \varepsilon_x \cos^2\theta_a + \varepsilon_y \sin^2\theta_a + \gamma_{xy} \sin\theta_a \cos\theta_a$$
$$\varepsilon_b = \varepsilon_x \cos^2\theta_b + \varepsilon_y \sin^2\theta_b + \gamma_{xy} \sin\theta_b \cos\theta_b$$
$$\varepsilon_c = \varepsilon_x \cos^2\theta_c + \varepsilon_y \sin^2\theta_c + \gamma_{xy} \sin\theta_c \cos\theta_c$$

For the 45^o strain rosette as shown in Fig. 1.32b,

$$\varepsilon_x = \varepsilon_a$$
$$\varepsilon_y = \varepsilon_c$$
$$\gamma_{xy} = 2\varepsilon_b - (\varepsilon_a + \varepsilon_c)$$

For the 60^o strain rosette as shown in Fig. 1.32c,

$$\varepsilon_x = \varepsilon_a$$
$$\varepsilon_y = \frac{1}{3}(2\varepsilon_b + 2\varepsilon_c - \varepsilon_a)$$
$$\gamma_{xy} = \frac{2}{\sqrt{3}}(\varepsilon_b - \varepsilon_c)$$

Example 1-4

The state of strains at point A on the bracket as shown in Fig. Ex 1-4 is measured using the stain rosette as shown. Due to the loadings, the readings from the gauge give $\varepsilon_a = 60\,\mu\varepsilon$, $\varepsilon_b = 135\,\mu\varepsilon$, and $\varepsilon_c = 264\,\mu\varepsilon$. Determine the in-plane principal strains at the point and the directions in which they act.

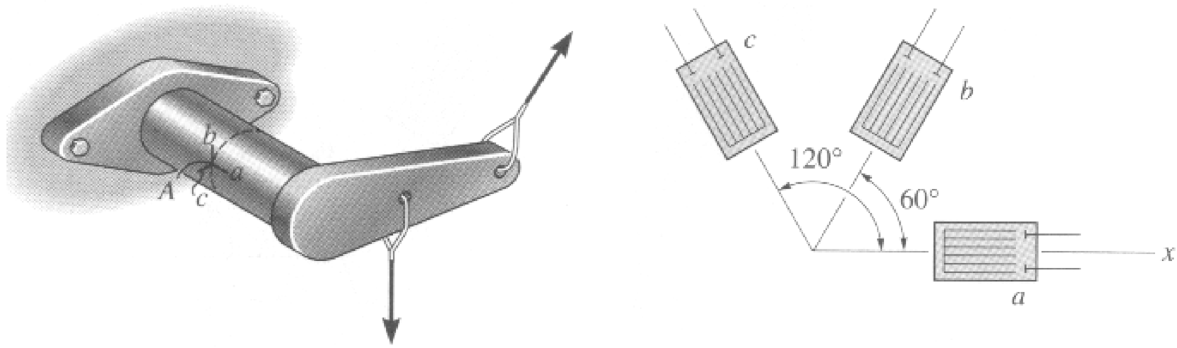


Fig. Ex 1-4

Setting the $+x$ axis as shown and measuring the angles counterclockwise from the $+x$ axis to the center-lines of each gauge. We have

$$\theta_a = 0^o,\ \theta_b = 60^o, \text{ and } \theta_c = 120^o$$

Therefore, we obtain

$$60(10^{-6}) = \varepsilon_x \cos^2 0^o + \varepsilon_y \sin^2 0^o + \gamma_{xy} \sin 0^o \cos 0^o$$

$$\varepsilon_x = 60(10^{-6})$$

$$135(10^{-6}) = \varepsilon_x \cos^2 60^o + \varepsilon_y \sin^2 60^o + \gamma_{xy} \sin 60^o \cos 60^o$$

$$0.25\varepsilon_x + 0.75\varepsilon_y + 0.433\gamma_{xy} = 135(10^{-6})$$

$$264(10^{-6}) = \varepsilon_x \cos^2 120^o + \varepsilon_y \sin^2 120^o + \gamma_{xy} \sin 120^o \cos 120^o$$

$$0.25\varepsilon_x + 0.75\varepsilon_y - 0.433\gamma_{xy} = 264(10^{-6})$$

Solving the simultaneous equations, we obtain

$$\varepsilon_x = 60(10^{-6}),\ \varepsilon_y = 246(10^{-6}), \text{ and } \gamma_{xy} = -149(10^{-6})$$

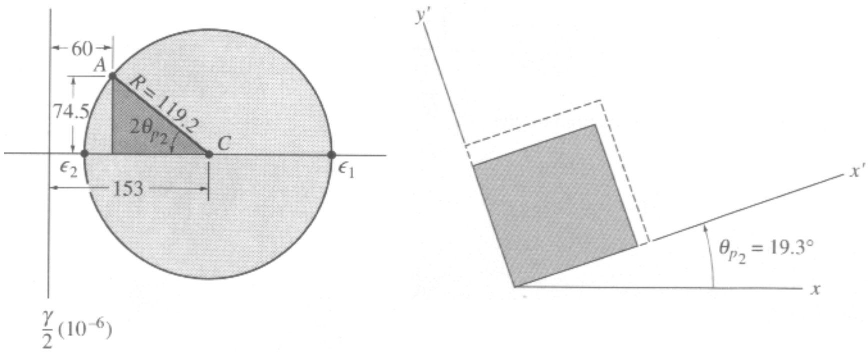
The in-plane principal strains and their directions

The in-plane principal strains and their directions can be determined by the equations

$$\varepsilon_{\substack{1\\2}} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

or by using the Mohr's circle as shown below.



Hence, the in-plane principal strains and their directions are

$$\varepsilon_1 = 272(10^{-6})$$

$$\varepsilon_2 = 33.8(10^{-6})$$

$$2\theta_{p2} = \tan^{-1}\frac{74.5}{153-60} = 38.7^o$$

$$\theta_{p2} = 19.3^o$$

The state of the in-plane principal strains is shown in the figure above. The dashed line shows the deformed configuration of the element.

1.9 Small-Displacement Theory

The derivation of the strains in the previous sections is purely based on the geometrical consideration and the obtained equations are exact. However, they are highly nonlinear partial differential equations that are difficult to solve. In practice, the displacements are usually small compared with the dimensions of the body, thus, the squares and the products of the strains and their first derivatives are infinitesimal small quantities. By using this fact, we can simplify the analysis of the deformable body significantly.

If the displacements and their derivatives are small,

1. The strains of fibers in one plane are not influenced by the out-of-plane displacements.
2. The undeformed geometry of the body can be used when writing the equilibrium equations
3. The stress-strain relations are reduced to linear relations.

Consider the displacement and deformation in the $x-y$ plane moving from point 1, 0, and 2 to the locations $1'$, $0'$, and $2'$, respectively, as shown in Fig. 1.33.

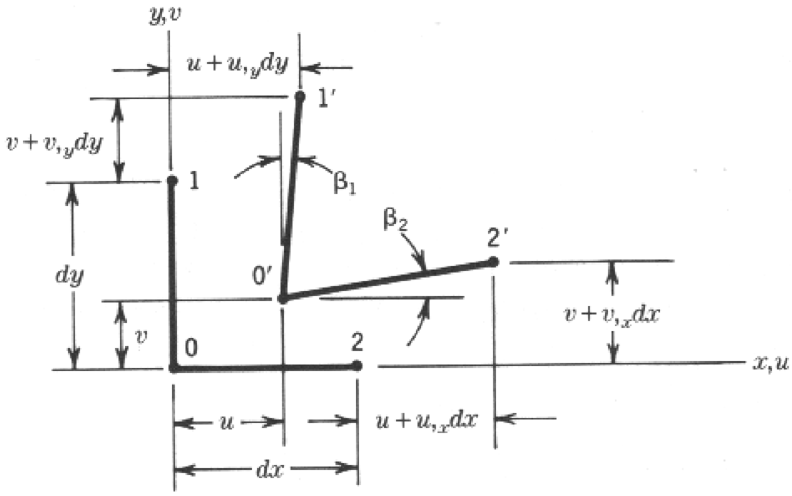


Fig. 1.33

By the definition of normal strain,

$$\varepsilon_x = \frac{L_{0'2'} - L_{02}}{L_{02}} = \frac{[dx + (u + u,_x\, dx) - u] - dx}{dx} = u,_x = \frac{\partial u}{\partial x}$$

Similarly,

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

For small displacement analysis, the engineering shear strain

$$\gamma_{xy} = \beta_1 + \beta_2 = \frac{(u + u,_y\, dy) - u}{dy} + \frac{(v + v,_x\, dx) - v}{dx} = u,_y + v,_x = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Similarly, if we consider the displacement and deformation in the $y-z$ and $x-z$ plane, we have

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

In this form of strain.displacement relations, the physical interpretation of the strain components can be seen clearly. For example, the normal strain ε_x is the rate of change of the displacement u with respect to x. The shearing strain γ_{xy} represents the changes in the original right angles between the line elements 0-1, and 0-2 to the $0'$-$1'$ and $0'$-$2'$ due to the deformation.

Strain compatibility Relations

Just as stresses must satisfy the equations of equilibrium, the strains must satisfy the strain compatibility equations in order to describe a physically possible displacement field. The displacement field must be single-valued, continuous, and has continuous derivatives. Thus, the material does not overlap itself and no crack appears.

By eliminating the displacement components from the strains equations, we have

$$\frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2 \gamma_{xy}}{\partial z^2} = \frac{1}{2}\frac{\partial^2 \gamma_{yz}}{\partial z \partial x} + \frac{1}{2}\frac{\partial^2 \gamma_{zx}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_y}{\partial x \partial z} + \frac{1}{2}\frac{\partial^2 \gamma_{xz}}{\partial y^2} = \frac{1}{2}\frac{\partial^2 \gamma_{xy}}{\partial y \partial z} + \frac{1}{2}\frac{\partial^2 \gamma_{yz}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_x}{\partial y \partial z} + \frac{1}{2}\frac{\partial^2 \gamma_{yz}}{\partial x^2} = \frac{1}{2}\frac{\partial^2 \gamma_{xz}}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2 \gamma_{xy}}{\partial x \partial z}$$

Example 1-5

The parallelepiped as shown in Fig. Ex 1-5 is deformed into the shape indicated by the dashed straight line (small displacements). The displacements are given by the following relations: $u = C_1 xyz$, $v = C_2 xyz$, and $w = C_3 xyz$.

a.) Determine the state of strain at point E when the coordinate of point E^* for the deformed body are (1.504, 1.002, 1.996).

b.) Check if the state of strain as point E is in accordance with the strain compatibility relations.

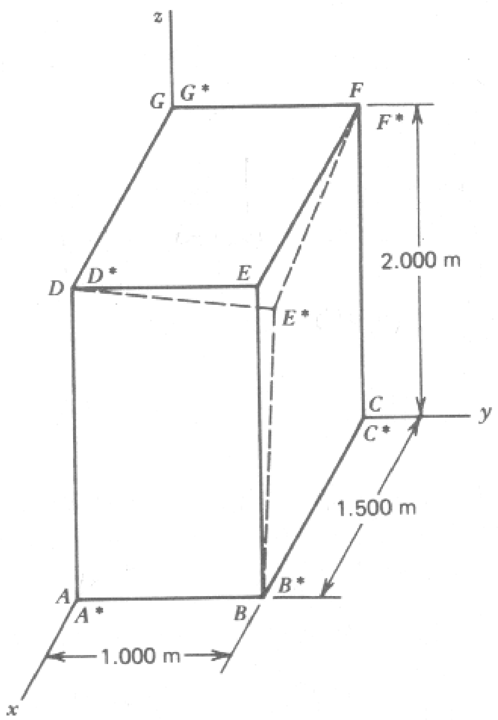


Fig. Ex 1-5

The state of strain as point E

The displacements of point E are $u = 1.504 - 1.5 = 0.004\text{ m}$, $v = 1.002 - 1 = 0.002\text{ m}$, and $w = 1.996 - 2 = -0.004\text{ m}$. Thus, the displacement relations are in the form of

$$C_1 = \frac{u}{xyz} = \frac{0.004}{1.5(1)2} = \frac{0.004}{3}$$

$$u = \frac{0.004}{3} xyz$$

In the same manner, we have

$$v = \frac{0.002}{3} xyz$$

$$w = -\frac{0.004}{3} xyz.$$

Thus, the strains at point E are

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{0.004}{3} yz = \frac{0.004}{3}(1)2 = 0.00267$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{0.002}{3} xz = \frac{0.002}{3}(1.5)2 = 0.00200$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = -\frac{0.004}{3} xy = -0.00200$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{0002}{3} yz + \frac{0.004}{3} xz = 0.00533$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\frac{0.004}{3} yz + \frac{0.004}{3} xy = -0.00067$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{0.002}{3} xy - \frac{0.004}{3} xz = -0.00300$$

Substituting the strain equations into the strain compatibility relations, we can see that the state of strain as point E is in accordance with the strain compatibility relations.

Chapter 2

Stress and Strain Relations

2.1 Concept of Engineering Stress-Strain and True Stress-Strain

Engineering stress-strain behavior is usually determined from monotonic tension test. Fig. 2.1 shows the free body diagram of the test specimen. Originally, the specimen has the cross-sectional area of A_o and the length of l_o. Under the action of the axial tensile load P, the test specimen has the cross-sectional area of A and the length of l.

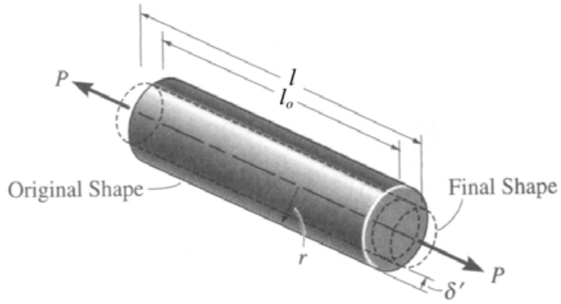


Fig. 2.1

Engineering stress $$\sigma = \frac{P}{A_o}$$

Engineering strain $$\varepsilon = \frac{l - l_o}{l_o} = \frac{\delta}{l_o}$$

True stress $$\tilde{\sigma} = \frac{P}{A}$$

True strain $$\tilde{\varepsilon} = \int_{l_o}^{l} \frac{1}{l} dl = \ln \frac{l}{l_o}$$

The use of the true stress and true strain changes the appearance of the monotonic tension stress-strain curve as shown in Fig. 2.2.

From the engineering stress-strain diagram, the material behavior can be classified into 4 different ways depending on how the material behave.

Elastic behavior (1st region)

The specimen is called to response *elastically* if it returns to its original shape or length after the load acting on it is removed.

In this region, the stress is proportional to the strain from the origin to the *proportional limit* (stress has a linear relationship with strain).

If the stress is slightly over to the proportional limit, the material may still respond elastically. If the stress is increased gradually, the slope of the curve tends to get smaller and smaller until it reaches the elastic limit.

Yielding (2nd region)

This region is started when the *yielding stress* is reached. The yielding stress is the stress which the material starts to deform permanently. After this point, the specimen will continue to elongate without any increase in load, *perfectly plastic*.

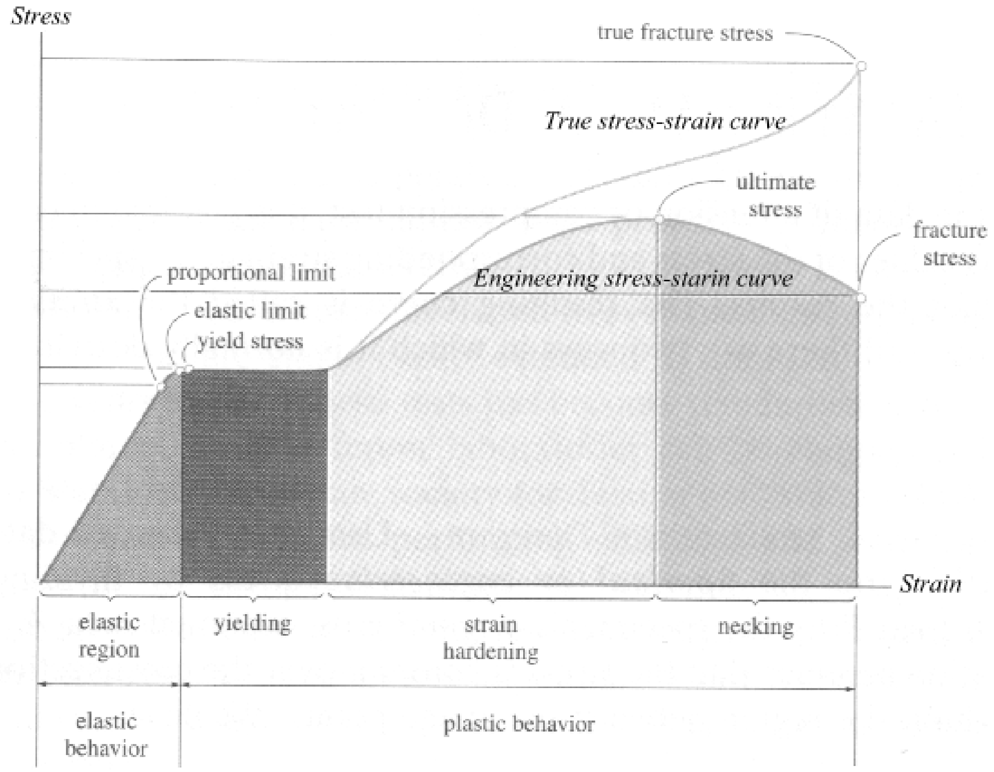


Fig. 2.2

Strain Hardening (3th region)

When the yielding has ended, a further load increase can be applied to the specimen, resulting in a curve that rise continuously but becomes flatter until it reaches the *ultimate stress*. The rise in the curve is called *strain hardening*.

Necking (4th region)

At the ultimate stress, the cross-sectional area begins to decrease in a localized region of the specimen, *necking*. This phenomenon is caused by slip planes of randomly oriented crystals formed within the material. As the cross-sectional area is continually decreased, the load is also gradually decreased, resulting in the stress strain diagram tends to curve downward until the specimen breaks at the *fracture stress*.

If a specimen made of ductile material, such as steel and brass, is loaded pass the yield point A to the plastic region at point A' and then unloaded, elastic strain is recovered to point O' as the material return to its equilibrium as shown in Fig. 2.3. But, the plastic strain remains. As a result, the material is subjected to a *permanent set*.

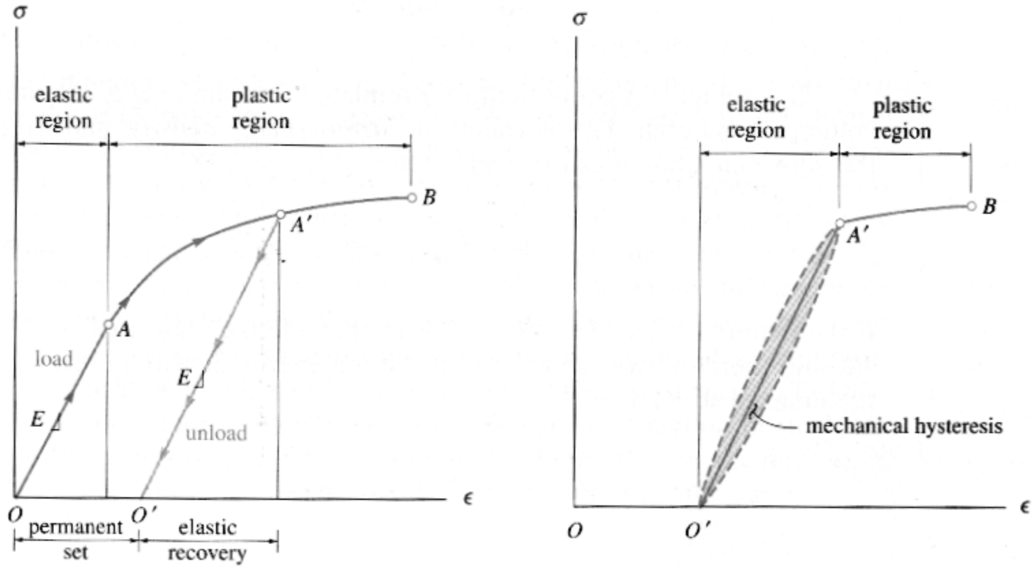


Fig. 2.3

In general, some heat and energy may be lost as the specimen is unloaded from A' to O' and loaded again from O' to A'. As a result, the unloaded and loaded curves will have the shape as shown. The colored area between these curves represents the energy lost and called *hysteresis loop*.

True stress-Strain Curve

The true stress is always larger than the corresponding engineering stress, and the difference may be of a factor of two or more near the end of a tensile test on a ductile material. True strain based on a length measurement is somewhat smaller than the corresponding engineering strain. However, once the necking starts, true strain based on an area measurement are larger.

The total true strain in a tension test can be separated into two components.

1. Linearly elastic strain $\tilde{\varepsilon}_e$ that can be recovered upon unloading.
2. Nonlinearly plastic strain $\tilde{\varepsilon}_p$ that can not be recovered upon unloading.

$$\tilde{\varepsilon}_{total} = \tilde{\varepsilon}_e + \tilde{\varepsilon}_p$$

Consider the true stress-strain curve of a metal in the region well beyond yielding, where most strain is plastic strain. A logarithmic plot of true stress versus true strain in this region gives a straight line as shown in Example 2-1 and the true stress-true strain relationship is in the form of

$$\tilde{\sigma}_{total} = K(\tilde{\varepsilon}_p)^n$$

where $n =$ strain hardening coefficient

$K =$ strength coefficient

True fracture strength,

$$\tilde{\sigma}_f = \frac{P_f}{A_f}$$

True fracture ductility

$$\tilde{\varepsilon}_f = \ln\frac{l_f}{l_o}$$

Volume of the material of the specimen is $l_o A_o = l_f A_f$. Then,

$$\frac{A_o}{A_f} = \frac{l_f}{l_o}$$

$$\tilde{\varepsilon}_f = \ln\frac{A_o}{A_f}$$

Let $RA =$ area reduction of the specimen $= \dfrac{A_o - A_f}{A_o}$. Hence,

$$\frac{1}{1-RA} = \frac{A_o}{A_f} = \frac{l_f}{l_o}$$

$$\tilde{\varepsilon}_f = \ln\frac{1}{1-RA}$$

Since $\tilde{\sigma} = K(\tilde{e}_p)^n$, at failure,

$$\tilde{\sigma}_f = K(\tilde{e}_p^{failure})^n = K(\tilde{e}_f)^n$$

$$K = \frac{\tilde{\sigma}_f}{(\tilde{\varepsilon}_f)^n}$$

where $\tilde{\varepsilon}_f = \tilde{\varepsilon}_e^{failure} + \tilde{\varepsilon}_p^{failure} \approx \tilde{\varepsilon}_p^{failure}$.

Since $\tilde{\sigma} = K(\tilde{\varepsilon}_p)^n$, $\tilde{\varepsilon}_p = \left(\dfrac{\tilde{\sigma}}{K}\right)^{\frac{1}{n}}$. Then,

$$\tilde{\varepsilon}_p = \left(\frac{\tilde{\sigma}}{\dfrac{\tilde{\sigma}_f}{(\tilde{\varepsilon}_f)^n}}\right)^{\frac{1}{n}} = \left(\frac{\tilde{\sigma}}{\tilde{\sigma}_f}\right)^{\frac{1}{n}} \tilde{\varepsilon}_f$$

If n, $\tilde{\varepsilon}_f$, and $\tilde{\sigma}_f$ are known, $\tilde{\varepsilon}_p$ at a given $\tilde{\sigma}$ can be determined. Also,

$$\tilde{\varepsilon}_{total} = \tilde{\varepsilon}_e + \tilde{\varepsilon}_p = \frac{\tilde{\sigma}}{E} + \left(\frac{\tilde{\sigma}}{\tilde{\sigma}_f}\right)^{\frac{1}{n}} \tilde{\varepsilon}_f$$

$$\tilde{\varepsilon}_{f,total} = \frac{\tilde{\sigma}_f}{E} + \tilde{\varepsilon}_p^{failure}$$

Plastic strain at necking

Plastic strain at necking is equal to the magnitude of strain hardening coefficient for a given material.

$$n = \tilde{\varepsilon}_p^{necking}$$

<u>Proof</u>

Since the elastic strain is small compared to the plastic strain, we can neglect the elastic strain.

$$\tilde{\sigma} = \frac{P}{A} = \frac{P}{A_o}\frac{A_o}{A} = \sigma\frac{A_o}{A}$$

$$\varepsilon = \frac{l - l_o}{l_o} = \frac{l}{l_o} - 1$$

Since $\frac{l}{l_o} = \frac{A_o}{A}$, thus, $\varepsilon = \frac{A_o}{A} - 1$ and $\frac{A_o}{A} = 1 + \varepsilon$. The true stress is

$$\tilde{\sigma} = \sigma(1 + \varepsilon)$$

Since the nonlinearly elastic strain can be determined from $\tilde{\varepsilon}_p = \ln\frac{l}{l_o} = \ln(1+\varepsilon)$. Substituting the true stress $\tilde{\sigma} = \sigma(1+\varepsilon)$ and $\tilde{\varepsilon}_p$ into the equation $\tilde{\sigma} = K(\tilde{e}_p)^n$, we have

$$\sigma = \frac{K}{1+\varepsilon}[\ln(1+\varepsilon)]^n$$

At the necking point, the slope of the engineering stress-strain curve is equal to zero.

$$\frac{d\sigma}{d\varepsilon}\Big|_{necking} = \frac{K}{(1+\varepsilon)^{-2}}[\ln(1+\varepsilon)]^n\left[-1 + \frac{n}{\ln(1+\varepsilon)}\right]\Big|_{necking} = 0$$

Since the 1^{st} and 2^{nd} term can not be zero, thus, the 3^{rd} term must be zero.

$$n = \ln(1+\varepsilon)\Big|_{necking} = \tilde{\varepsilon}_p^{necking}$$

Bridgeman correction for hoop stress

A complication exists in interpreting tensile results near the end of a test where there is a large amount of necking. Bridgeman in 1944 pointed out that large amounts of necking result in a tensile hoop stress being generated around the circumference in the necked region. Thus, the state of stress is no longer uniaxial as assumed, and the behavior of the material is affected. In particular, the axial stress is increased above what it would otherwise be. The corrected value of true stress can be calculated from

$$\tilde{\sigma}_B = B\tilde{\sigma}$$

where

$$B = 0.83 - 0.186\log\tilde{\varepsilon} \qquad (0.15 \le \tilde{\varepsilon} \le 3)$$

Example 2-1

A tension test was conducted on a specimen of AISI 1020 hot-rolled steel having an initial diameter of 9.11 mm. The test data is given in Table Ex 2-1 where the length changes over a 50 mm gage length have been converted to engineering strain ε in the first column. Loads at corresponding times are given in the second column. Also, diameters for the large strain portion of the test, measured in the neck when the necking once started, are given in the third column. After fracture, the gage length had stretched to 68.5 mm.

Table Ex 2-1

Test Data			Calculated Values			
Engr. Strain ε	Load P kN	Diameter d mm	Engineering Stress σ MPa	True Strain[5] $\tilde{\varepsilon}$	Raw True Stress[5] $\tilde{\sigma}$ MPa	Corrected True Stress $\tilde{\sigma}_B$ MPa
0	0	9.11	0	0	0	0
0.0015[1]	19.13	—	293	0.00150	293	—
0.0033[2]	17.21	—	264	0.00329	265	—
0.0050	17.53	—	269	0.00499	270	—
0.0070	17.44	—	268	0.00698	269	—
0.010	17.21	—	264	0.00995	267	—
0.049	20.77	8.89	319	0.0489	335	335
0.218	25.71	8.26	394	0.196	480	461
0.234[3]	25.75	—	395	0.210	488	466
0.306	25.04	7.62	384	0.357	549	501
0.330	23.49	6.99	360	0.530	612	539
0.348	21.35	6.35	328	0.722	674	577
0.360	18.90	5.72	290	0.931	735	615
0.366[4]	17.39	5.28[6]	267	1.091	794	654

Notes: [1]Upper yield. [2]Lower yield and 0.2% offset yield. [3]Ultimate. [4]Fracture. [5]Calculated from $(1+\varepsilon)$ where d not measured. [6]Measured from the broken specimen.

a.) Plot the engineering stress-strain diagram.

b.) Determine the yielding strength, ultimate tensile strength, percent elongation, percent reduction of area, and the modulus of toughness.

c.) Plot the true stress-strain diagram.

d.) Determine the true stress-true strain relationship in the plastic region.

The engineering stress-strain diagram can be plotted by finding the engineering stress from the equation $\sigma = \dfrac{P}{A_o}$ as shown in the forth column of the table. From the data of the engineering stress and engineering strain, we can plot the engineering stress-strain curve as shown in Fig Ex 2-1a.

The load for the 0.2% percent offset yield, corresponding to the lower yield point, is 17.21 kN. Thus, the yielding strength is

$$\sigma_y = \frac{17210}{\pi(0.00911)^2/4} = 264\,\text{MPa}$$

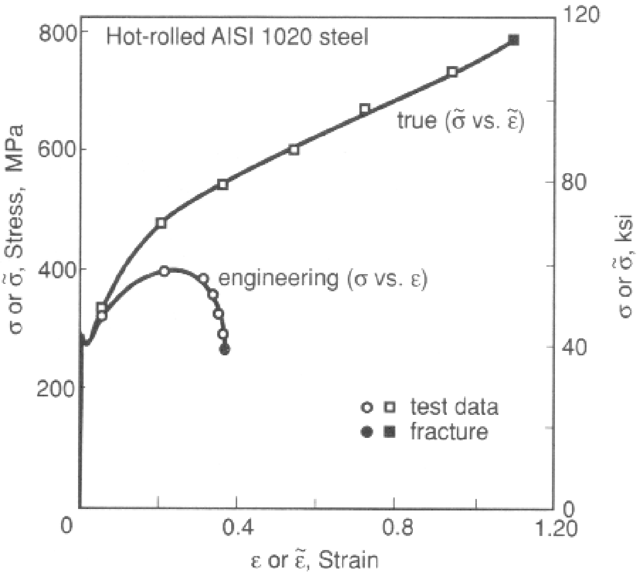


Fig Ex 2-1a

The highest load reached was 25.75 kN. Thus, the ultimate tensile strength is

$$\sigma_u = \frac{25750}{\pi(0.00911)^2/4} = 395\,\text{MPa}$$

The percent elongation for 50 mm gage length is

$$100\frac{L_f - L_i}{L_i} = 100\frac{68.5-50}{50} = 37\%$$

The final diameter was 5.28 mm. Thus, the percent reduction of area is

$$100\frac{d_i^2 - d_f^2}{d_i^2} = 100\frac{9.11^2 - 5.28^2}{9.11^2} = 66.4\%$$

The modulus of toughness of AISI 1020 hot-rolled steel can be estimated by the equation

$$u_f \approx \varepsilon_f\left[\frac{\sigma_y + \sigma_u}{2}\right] = 0.366\left[\frac{264+395}{2}\right] = 120\,\frac{\text{MJ}}{\text{m}^3}$$

Since we do not have the data of the length and diameter of the specimen up to the engineering strain 0.010. Therefore, the true stress-strain diagram in this portion must be plotted by finding the true stress and true strain from the equation

$$\tilde{\sigma} = \sigma(1+\varepsilon)$$

$$\tilde{\varepsilon} = \ln(1+\varepsilon)$$

After that the true stress-strain diagram can be plotted by finding the true stress and true strain from the equation

$$\tilde{\sigma} = \frac{P}{A}$$

$$\tilde{\varepsilon} = \ln\frac{A_o}{A}$$

From the data of the true stress and true strain, we can plot the true stress-strain curve as shown in Fig Ex 2-1a.

The true stress-true strain relationship in the plastic region is in the form of

$$\tilde{\sigma} = K(\tilde{e}_p)^n$$

or

$$\log\tilde{\sigma} = n\log\tilde{\varepsilon}_p + \log K$$

which gives a straight line for the log-log true stress and true strain plot as shown in Fig Ex 2-1b. It should be noted that the true stress used in this plot is the true stress corrected by using the Bridgeman correction factor B.

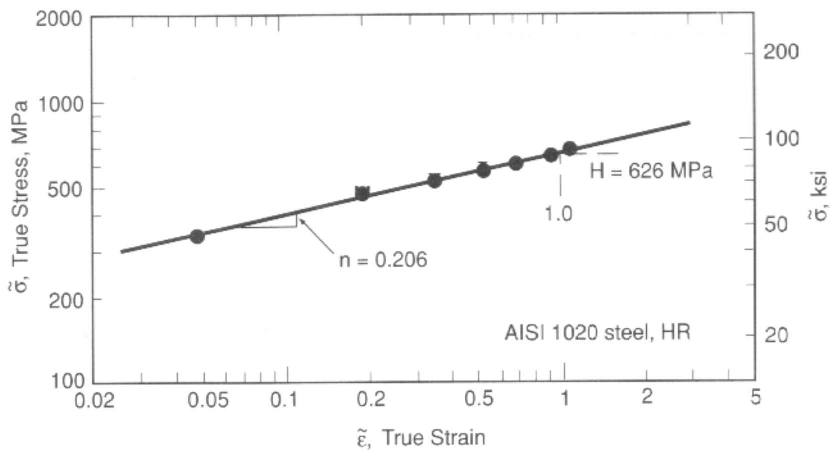


Fig Ex 2-1b

The equation is in a form of a straight line on an $x - y$ plot

$$y = mx + b$$

From the plot, we can see that the straight line will have a slope of n and intercept at $\tilde{\varepsilon}_p = 1$ of $\tilde{\sigma} = K$. By using a graphical or least square method, we obtain

$$m = n = 0.206$$

$$b = 2.7967$$

Since $b = \log K$, thus, $K = 10^b = 626\,\text{MPa}$ and the true stress-true strain relationship in the plastic region is in the form of

$$\tilde{\sigma} = 626(\tilde{e}_p)^{0.206}$$

It should be noted that $n \approx \tilde{\varepsilon}_p^{necking} = 0.210$.

2.2 First Law of Thermodynamics. Internal Energy Density. Complementary Internal Energy Density

The stress-strain relations can be proved theoretically by using the first law of thermodynamics where we can show that the relations are symmetry. However, the elastic coefficients or stiffness coefficients of these relations are obtained experimentally.

The variations of the strain components resulting from the variations of the displacement δu, δv, and δw are

$$\delta\varepsilon_x = \frac{\partial(\delta u)}{\partial x} \qquad \delta\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left[\frac{\partial(\delta v)}{\partial x} + \frac{\partial(\delta u)}{\partial y}\right]$$

$$\delta\varepsilon_y = \frac{\partial(\delta v)}{\partial y} \qquad \delta\varepsilon_{yz} = \frac{1}{2}\gamma_{yz} = \frac{1}{2}\left[\frac{\partial(\delta w)}{\partial y} + \frac{\partial(\delta v)}{\partial z}\right]$$

$$\delta\varepsilon_z = \frac{\partial(\delta w)}{\partial z} \qquad \delta\varepsilon_{xz} = \frac{1}{2}\gamma_{xz} = \frac{1}{2}\left[\frac{\partial(\delta w)}{\partial x} + \frac{\partial(\delta u)}{\partial z}\right]$$

First Law of Thermodynamics

Consider the free body diagram of the body. The body has a volume of V^* and the body forces B_x, B_y, and B_z per unit volume as shown in Fig. 2.4. Under the action of the surface forces and during the displacement variations δu, δv, and δw, the body is in static equilibrium.

For a condition which no net heat flow into the volume V^*, the first law of thermodynamic states that *during the displacement variations* δu, δv, and δw*, the variation in work of the external forces* δW_e *is equal to the variation of the internal energy* δU.

$$\delta W_e = \delta U$$

$$\delta W_S + \delta W_B = \delta U$$

where $\delta W_S =$ the work of the surface forces and $\delta W_B =$ the work of the body forces.

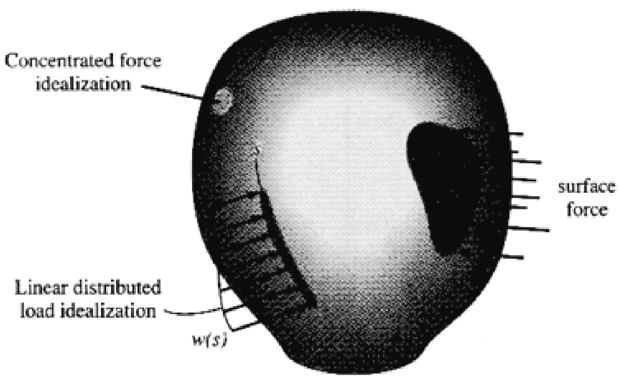


Fig. 2.4

From the previous chapter, the components of the stress vector $\vec{\sigma}_P = \sigma_{Px}\hat{i} + \sigma_{Py}\hat{j} + \sigma_{Pz}\hat{k}$ acting on the plane P having the normal vector $\vec{n} = l\hat{i} + m\hat{j} + n\hat{k}$ as shown in Fig. 2.5 can be written as

$$\sigma_{Px} = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$

$$\sigma_{Py} = \tau_{xy} l + \sigma_y m + \tau_{yz} n$$

$$\sigma_{Pz} = \tau_{xz} l + \tau_{yz} m + \sigma_z n$$

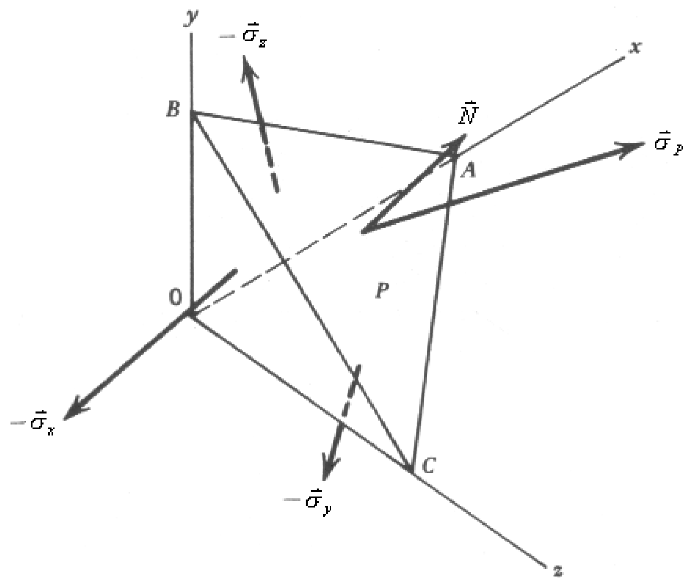


Fig. 2.5

The force components due the stress vector components acting on the area $dS = dA_{ABC}$ are

$$dF_x = \sigma_{Px} dS \qquad dF_y = \sigma_{Py} dS \qquad dF_z = \sigma_{Pz} dS$$

The work δW_S can be written as

$$\delta W_S = \iint_{S^*} \delta u(\sigma_{Px} dS) + \iint_{S^*} \delta v(\sigma_{Py} dS) + \iint_{S^*} \delta w(\sigma_{Pz} dS)$$

$$\delta W_S = \iint_{S^*} [\sigma_x l + \tau_{xy} m + \tau_{xz} n)\delta u + (\tau_{xy} l + \sigma_y m + \tau_{yz} n)\delta v + (\tau_{xz} l + \tau_{yz} m + \sigma_z n)\delta w] dS$$

The work δW_B can be written as

$$\delta W_B = \iiint_{V^*} (B_x \delta u + B_y \delta v + B_z \delta w) dV$$

The variation in work of the external forces δW_e is

$$\delta W_e = \iint_{S^*} [\sigma_x l + \tau_{xy} m + \tau_{xz} n)\delta u + (\tau_{xy} l + \sigma_y m + \tau_{yz} n)\delta v + (\tau_{xz} l + \tau_{yz} m + \sigma_z n)\delta w] dS +$$

$$\iiint_{V^*} (B_x \delta u + B_y \delta v + B_z \delta w) dV$$

Gauss's theorem

Let the vector on the surface dS having the total volume dV is $S_x\hat{i} + S_y\hat{j} + S_z\hat{k}$. The surface dS has the normal vector $\vec{n} = l\hat{i} + m\hat{j} + n\hat{k}$. By using the gauss's theorem, we have

$$\iiint_V \left[\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + \frac{\partial S_z}{\partial z}\right] dV = \iint_S (lS_x + mS_y + nS_z) dS$$

Rewriting the equation of the work δW_S into the form of $\iint_S (lS_x + mS_y + nS_z) dS$, we have

$$\delta W_S = \iint_{S^*} [(\sigma_x \delta u + \tau_{xy}\delta v + \tau_{xz}\delta w)l + (\tau_{xy}\delta u + \sigma_y \delta v + \tau_{yz}\delta w)m + (\tau_{xz}\delta u + \tau_{yz}\delta v + \sigma_z \delta w)n] dS$$

Let $S_x = \sigma_x \delta u + \tau_{xy}\delta v + \tau_{xz}\delta w$, $S_y = \tau_{xy}\delta u + \sigma_y \delta v + \tau_{yz}\delta w$, and $S_z = \tau_{xz}\delta u + \tau_{yz}\delta v + \sigma_z \delta w$, then, we have the work δW_S in the form of

$$\iiint_V \left[\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + \frac{\partial S_z}{\partial z}\right] dV = \iiint_{V^*} \left[\frac{\partial}{\partial x}(\sigma_x \delta u + \tau_{xy}\delta v + \tau_{xz}\delta w) + \frac{\partial}{\partial y}(\tau_{xy}\delta u + \sigma_y \delta v + \tau_{yz}\delta w) \right.$$
$$\left. + \frac{\partial}{\partial z}(\tau_{xz}\delta u + \tau_{yz}\delta v + \sigma_z \delta w)\right] dV$$

The variation in work of the external forces δW_e can be rewritten in the form of

$$\delta W_e = \iiint_{V^*} \left[\frac{\partial}{\partial x}(\sigma_x \delta u + \tau_{xy}\delta v + \tau_{xz}\delta w) + \frac{\partial}{\partial y}(\tau_{xy}\delta u + \sigma_y \delta v + \tau_{yz}\delta w) \right.$$
$$\left. + \frac{\partial}{\partial z}(\tau_{xz}\delta u + \tau_{yz}\delta v + \sigma_z \delta w) + B_x \delta u + B_y \delta v + B_z \delta w\right] dV$$

Note that the partial derivatives,

$$\frac{\partial}{\partial x}\sigma_x \delta u = \sigma_x \frac{\partial \delta u}{\partial x} + \delta u \frac{\partial \sigma_x}{\partial x} = \sigma_x \delta \varepsilon_x + \delta u \frac{\partial \sigma_x}{\partial x}$$

$$\frac{\partial}{\partial x}\tau_{xy}\delta v = \tau_{xy}\frac{\partial \delta v}{\partial x} + \delta v \frac{\partial \tau_{xy}}{\partial x}$$

$$\frac{\partial}{\partial x}\tau_{xz}\delta w = \tau_{xz}\frac{\partial \delta w}{\partial x} + \delta w \frac{\partial \tau_{xz}}{\partial x}$$

$$\frac{\partial}{\partial y}\tau_{xy}\delta u = \tau_{xy}\frac{\partial \delta u}{\partial y} + \delta u \frac{\partial \tau_{xy}}{\partial y}$$

$$\frac{\partial}{\partial y}\sigma_y \delta v = \sigma_y \frac{\partial \delta v}{\partial y} + \delta v \frac{\partial \sigma_y}{\partial y} = \sigma_y \delta \varepsilon_y + \delta v \frac{\partial \sigma_y}{\partial y}$$

$$\vdots$$

$$\frac{\partial}{\partial z}\sigma_z \delta w = \sigma_z \frac{\partial \delta w}{\partial z} + \delta w \frac{\partial \sigma_z}{\partial z} = \sigma_z \delta \varepsilon_z + \delta w \frac{\partial \sigma_z}{\partial z}$$

Rearranging the terms, we have

$$\delta W_e = \iiint_{V^*} (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + 2\tau_{xy}\delta\varepsilon_{xy} + 2\tau_{yz}\delta\varepsilon_{yz} + 2\tau_{xz}\delta\varepsilon_{xz})dV +$$

$$\iiint_{V^*} \left[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x \right)\delta u + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + B_y \right)\delta v \right.$$

$$\left. + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + B_z \right)\delta w \right] dV$$

Since the components in the second integral is the differential equilibrium equation of deformable body which is equal to zero, then,

$$\delta W_e = \iiint_{V^*} (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + 2\tau_{xy}\delta\varepsilon_{xy} + 2\tau_{yz}\delta\varepsilon_{yz} + 2\tau_{xz}\delta\varepsilon_{xz})dV$$

or,

$$\delta W_e = \iiint_{V^*} (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz})dV$$

The internal energy U of the volume V^* of the body can be expressed in terms of the internal-energy density U_o as

$$U = \iiint_{V^*} U_o dV$$

The variation of the internal energy δU is

$$\delta U = \iiint_{V^*} \delta U_o dV$$

Since the first law of thermodynamic states that

$$\delta W_e = \delta U = \iiint_{V^*} (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + 2\tau_{xy}\delta\varepsilon_{xy} + 2\tau_{yz}\delta\varepsilon_{yz} + 2\tau_{xz}\delta\varepsilon_{xz})dV$$

The variation of the internal-energy density δU_o is

$$\delta U_o = \sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + 2\tau_{xy}\delta\varepsilon_{xy} + 2\tau_{yz}\delta\varepsilon_{yz} + 2\tau_{xz}\delta\varepsilon_{xz}$$

In the index notation, we have

$$\delta U_o = \sigma_i \delta\varepsilon_i$$

$$i = 1, 2, 3, 4, 5, 6$$

Elasticity and internal energy density

For linearly elastic material, the total *internal energy* in a loaded body is equal to the total potential energy of the internal forces or *elastic strain energy*.

Since the strain-energy density U_o generally depends on the strain components, the coordinates (for inhomogeneous material), and the temperature. Thus, mathematically,

$$U_o = U_o(\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{xz}, x, y, z, T)$$

If the displacements u, v, and w have a variation δu, δv, and δw, respectively, the strain components will take the variations $\delta\varepsilon_x$, $\delta\varepsilon_y$, $\delta\varepsilon_z$, $\delta\varepsilon_{xy}$, $\delta\varepsilon_{yz}$, and $\delta\varepsilon_{xz}$. Therefore, the variation of the strain-energy density δU_o can be written as

$$\delta U_o = \frac{\partial U_o}{\partial \varepsilon_x}\,\delta\varepsilon_x + \frac{\partial U_o}{\partial \varepsilon_y}\,\delta\varepsilon_y + \frac{\partial U_o}{\partial \varepsilon_z}\,\delta\varepsilon_z + \frac{\partial U_o}{\partial \varepsilon_{xy}}\,\delta\varepsilon_{xy} + \frac{\partial U_o}{\partial \varepsilon_{yz}}\,\delta\varepsilon_{yz} + \frac{\partial U_o}{\partial \varepsilon_{xz}}\,\delta\varepsilon_{xz}$$

Comparing this equation with the one previously obtained $\delta U_o = \sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_z\delta\varepsilon_z + 2\tau_{xy}\delta\varepsilon_{xy} + 2\tau_{yz}\delta\varepsilon_{yz} + 2\tau_{xz}\delta\varepsilon_{xz}$, we have

$$\sigma_x = \frac{\partial U_o}{\partial \varepsilon_x} \qquad \sigma_y = \frac{\partial U_o}{\partial \varepsilon_y} \qquad \sigma_z = \frac{\partial U_o}{\partial \varepsilon_z}$$

$$\tau_{xy} = \frac{1}{2}\frac{\partial U_o}{\partial \varepsilon_{xy}} = \frac{\partial U_o}{\partial \gamma_{xy}} \quad \tau_{yz} = \frac{1}{2}\frac{\partial U_o}{\partial \varepsilon_{yz}} = \frac{\partial U_o}{\partial \gamma_{yz}} \quad \tau_{xz} = \frac{1}{2}\frac{\partial U_o}{\partial \varepsilon_{xz}} = \frac{\partial U_o}{\partial \gamma_{xz}}$$

In the matrix notation,

$$\sigma_i = \frac{\partial U_o}{\partial \varepsilon_i} \qquad\qquad i = 1, 2, 3, 4, 5, 6$$

Elasticity and complementary internal energy density

For structural members subjected to one component of stress such as in the simple tension test along the axis of the test specimen, the longitudinal stress σ_x can be written in the form of $\sigma_x = \dfrac{\partial U_o}{\partial \varepsilon_x}$. Thus, the strain-energy density in the specimen is

$$U_o = \int \sigma_x d\varepsilon_x$$

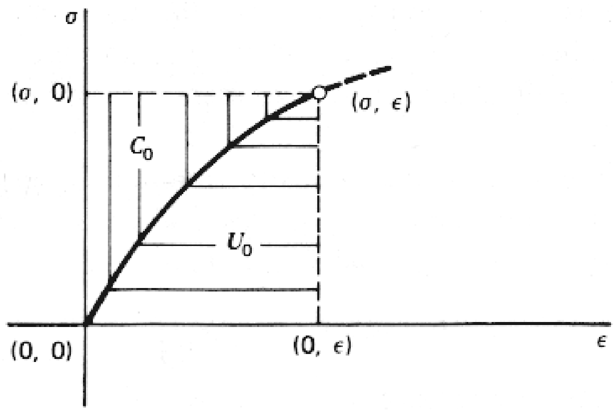


Fig. 2.6

This equation represents the area under the $\sigma_x - \varepsilon_x$ diagram as shown in Fig. 2.6. In addition, the total rectangular area is equal to

$$\sigma_x \varepsilon_x = U_o + C_o$$

where C_o is called the *complementary internal energy density* or *complementary strain energy density*. The C_o is represented by the area above the $\sigma_x - \varepsilon_x$ curve. Hence,

$$C_o = \int \varepsilon_x d\sigma_x$$

$$\varepsilon_x = \frac{dC_o}{d\sigma_x}$$

In general, the strain components are expressed as functions of the stress components.

$$\varepsilon_x = f_1(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$$

$$\varepsilon_y = f_2(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$$

$$\varepsilon_z = f_3(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$$

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = f_4(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$$

$$\varepsilon_{yz} = \frac{1}{2}\gamma_{yz} = f_5(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$$

$$\varepsilon_{xz} = \frac{1}{2}\gamma_{xz} = f_6(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$$

Thus, in analogous with the previous discussion, we have

$$C_o = -U_o + \sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + 2\tau_{xy}\varepsilon_{xy} + 2\tau_{yz}\varepsilon_{yz} + 2\tau_{xz}\varepsilon_{xz}$$

Differentiating the equation with respect to σ_x and using the chain rule, we have

$$\frac{\partial U_o}{\partial \sigma_x} = -\frac{\partial C_o}{\partial \sigma_x} + \left(\varepsilon_x \frac{\partial \sigma_x}{\partial \sigma_x} + \sigma_x \frac{\partial \varepsilon_x}{\partial \sigma_x}\right) + \left(\varepsilon_y \frac{\partial \sigma_y}{\partial \sigma_x} + \sigma_y \frac{\partial \varepsilon_y}{\partial \sigma_x}\right) + \left(\varepsilon_z \frac{\partial \sigma_z}{\partial \sigma_x} + \sigma_z \frac{\partial \varepsilon_z}{\partial \sigma_x}\right) +$$

$$2\left(\varepsilon_{xy} \frac{\partial \tau_{xy}}{\partial \sigma_x} + \tau_{xy} \frac{\partial \varepsilon_{xy}}{\partial \sigma_x}\right) + 2\left(\varepsilon_{yz} \frac{\partial \tau_{yz}}{\partial \sigma_x} + \tau_{yz} \frac{\partial \varepsilon_{yz}}{\partial \sigma_x}\right) + 2\left(\varepsilon_{xz} \frac{\partial \tau_{xz}}{\partial \sigma_x} + \tau_{xz} \frac{\partial \varepsilon_{xz}}{\partial \sigma_x}\right)$$

$$\frac{\partial U_o}{\partial \sigma_x} = -\frac{\partial C_o}{\partial \sigma_x} + \left(\varepsilon_x + \sigma_x \frac{\partial \varepsilon_x}{\partial \sigma_x}\right) + \left(\sigma_y \frac{\partial \varepsilon_y}{\partial \sigma_x}\right) + \left(\sigma_z \frac{\partial \varepsilon_z}{\partial \sigma_x}\right) + 2\left(\tau_{xy} \frac{\partial \varepsilon_{xy}}{\partial \sigma_x}\right) +$$

$$2\left(\tau_{yz} \frac{\partial \varepsilon_{yz}}{\partial \sigma_x}\right) + 2\left(\tau_{xz} \frac{\partial \varepsilon_{xz}}{\partial \sigma_x}\right)$$

Since $U_o = U_o(\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{xz})$ and $\varepsilon_x = f_1(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$, thus, by using the chain rule, we have

$$\frac{\partial U_o}{\partial \sigma_x} = \frac{\partial U_o}{\partial \varepsilon_x}\frac{\partial \varepsilon_x}{\partial \sigma_x} + \frac{\partial U_o}{\partial \varepsilon_y}\frac{\partial \varepsilon_y}{\partial \sigma_x} + \frac{\partial U_o}{\partial \varepsilon_z}\frac{\partial \varepsilon_z}{\partial \sigma_x} + \frac{\partial U_o}{\partial \varepsilon_{xy}}\frac{\partial \varepsilon_{xy}}{\partial \sigma_x} + \frac{\partial U_o}{\partial \varepsilon_{yz}}\frac{\partial \varepsilon_{yz}}{\partial \sigma_x} + \frac{\partial U_o}{\partial \varepsilon_{xz}}\frac{\partial \varepsilon_{xz}}{\partial \sigma_x}$$

$$\frac{\partial U_o}{\partial \sigma_x} = \sigma_x \frac{\partial \varepsilon_x}{\partial \sigma_x} + \sigma_y \frac{\partial \varepsilon_y}{\partial \sigma_x} + \sigma_z \frac{\partial \varepsilon_z}{\partial \sigma_x} + 2\tau_{xy} \frac{\partial \varepsilon_{xy}}{\partial \sigma_x} + 2\tau_{yz} \frac{\partial \varepsilon_{yz}}{\partial \sigma_x} + 2\tau_{xz} \frac{\partial \varepsilon_{xz}}{\partial \sigma_x}$$

Therefore, the terms in $\frac{\partial U_o}{\partial \sigma_x}$ will cancel some terms of the left-handed side of the previous equation, we have

$$\varepsilon_x = \frac{\partial C_o}{\partial \sigma_x}$$

Similarly, for differentiating the equation with respect to $\sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}$, and τ_{xz}, we obtain

$$\varepsilon_y = \frac{\partial C_o}{\partial \sigma_y} \qquad \varepsilon_z = \frac{\partial C_o}{\partial \sigma_z} \qquad \varepsilon_{xy} = \frac{1}{2}\frac{\partial C_o}{\partial \tau_{xy}} \quad \varepsilon_{yz} = \frac{1}{2}\frac{\partial C_o}{\partial \tau_{yz}} \quad \varepsilon_{xz} = \frac{1}{2}\frac{\partial C_o}{\partial \tau_{xz}}$$

For a linear elastic behavior, $C_o = U_o$ as shown in Fig. 2.7.

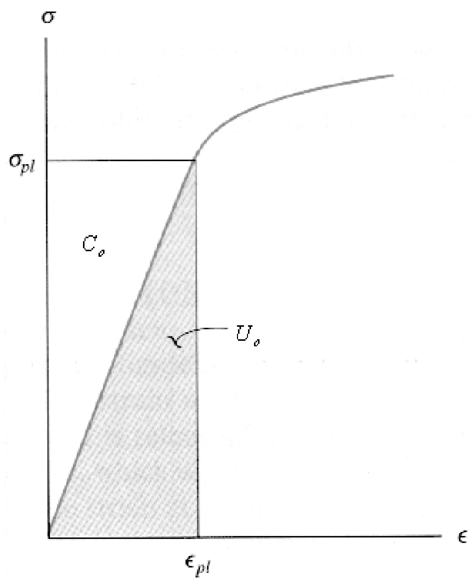


Fig. 2.7

2.3 Stress-Strain Relations and Strain-Stress Relations

The generalized Hooke's law relates stresses to strains. Each of the stress components is a linear function of the strain components.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

In the index notation, we have

$$\sigma_i = C_{ij}\varepsilon_j$$

$$i, j = 1, 2, 3, 4, 5, 6$$

or,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{Bmatrix}$$

The stiffness coefficients C_{ij} have 36 components. However, less than 36 of the coefficients can be shown actually independent for elastic material when the strain energy is considered. Using the relations between the stress components and the strain energy density, we have

$$\sigma_x = \frac{\partial U_o}{\partial \varepsilon_x} = C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\varepsilon_z + C_{14}\gamma_{yz} + C_{15}\gamma_{xz} + C_{16}\gamma_{xy}$$

$$\sigma_y = \frac{\partial U_o}{\partial \varepsilon_y} = C_{21}\varepsilon_x + C_{22}\varepsilon_y + C_{23}\varepsilon_z + C_{24}\gamma_{yz} + C_{25}\gamma_{xz} + C_{26}\gamma_{xy}$$

$$\sigma_z = \frac{\partial U_o}{\partial \varepsilon_z} = C_{31}\varepsilon_x + C_{32}\varepsilon_y + C_{33}\varepsilon_z + C_{34}\gamma_{yz} + C_{35}\gamma_{xz} + C_{36}\gamma_{xy}$$

$$\tau_{yz} = \frac{1}{2}\frac{\partial U_o}{\partial \varepsilon_{yz}} = \frac{\partial U_o}{\partial \gamma_{yz}} = C_{41}\varepsilon_x + C_{42}\varepsilon_y + C_{43}\varepsilon_z + C_{44}\gamma_{yz} + C_{45}\gamma_{xz} + C_{46}\gamma_{xy}$$

$$\tau_{xz} = \frac{1}{2}\frac{\partial U_o}{\partial \varepsilon_{xz}} = \frac{\partial U_o}{\partial \gamma_{xz}} = C_{51}\varepsilon_x + C_{52}\varepsilon_y + C_{53}\varepsilon_z + C_{54}\gamma_{yz} + C_{55}\gamma_{xz} + C_{56}\gamma_{xy}$$

$$\tau_{xy} = \frac{1}{2}\frac{\partial U_o}{\partial \varepsilon_{xy}} = \frac{\partial U_o}{\partial \gamma_{xy}} = C_{61}\varepsilon_x + C_{62}\varepsilon_y + C_{63}\varepsilon_z + C_{64}\gamma_{yz} + C_{65}\gamma_{xz} + C_{66}\gamma_{xy}$$

Hence, we can show that the stiffness coefficients are symmetry, $C_{ij} = C_{ji}$, by using appropriate differentiation.

$$\frac{\partial^2 U_o}{\partial \varepsilon_x \partial \varepsilon_y} = C_{12} = C_{21}$$

$$\frac{\partial^2 U_o}{\partial \varepsilon_x \partial \varepsilon_z} = C_{13} = C_{31}$$

$$\vdots$$

$$\frac{\partial^2 U_o}{\partial \varepsilon_x \partial \gamma_{xy}} = C_{16} = C_{61}$$

$$\vdots$$

$$\frac{\partial^2 U_o}{\partial \gamma_{xz} \partial \gamma_{yz}} = C_{56} = C_{65}$$

With the foregoing reduction from 36 to 21 components, the stress-strain relations are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

These relations are referred to as characterizing *anisotropic or triclinic materials* since there are no plane of symmetry for the material properties. Fig. 2.8 is a composite lamina in which the fiber direction $1-2$ has an angle with the loading or principal direction $x-y$. This composite material is an example of the anisotropic material.

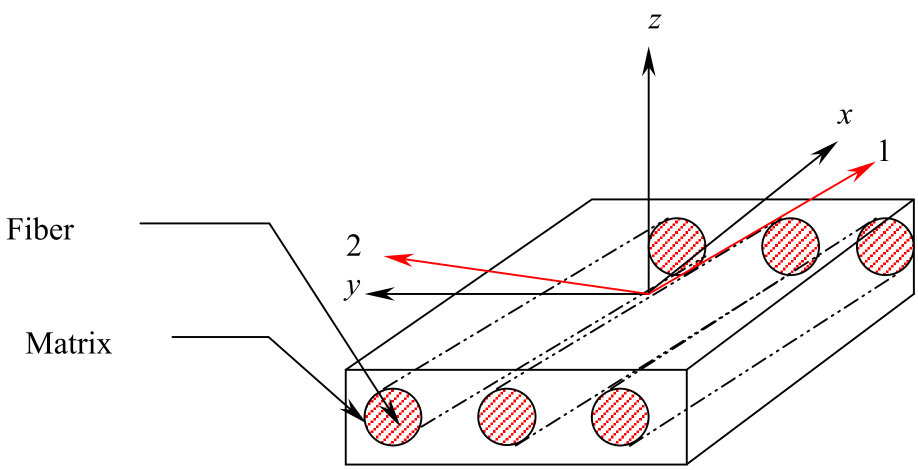


Fig. 2.8

If there is one plane of material property symmetry, the stress-strain relations reduce to

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

where the plane of symmetry is $z = 0$. Such a material is termed monoclinic *materials* having 13 independent stiffness coefficients.

If there are two orthogonal planes of material property symmetry for a material, symmetry will exist relative to a third mutually orthogonal plane. Then, the stress-strain relations reduce to

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

These stress-strain relations define *orthotropic materials*. There are 9 independent stiffness coefficients for an orthotropic material. Fig. 2.8 is a composite if the fiber direction $1-2$ coincides with the loading or principal direction $x-y$, this composite material is an example of the orthotropic material.

If there is one plane in the material in which the mechanical properties are equal in all direction, the material is called *transversely isotropic*. If the plane $x-y$ is the plane of isotropy, then, the subscript 1 and 2 of the stiffness coefficients are interchangeable. The stress-strain relations of the material will have 5 independent stiffness coefficients and are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

If the material is isotropic in which the mechanical properties of the material are symmetric on an infinite number of plane, there are only 2 independent stiffness coefficients and the stress-strain relations are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

Deformation characteristic

Let us consider Fig. 2.9. The deformation of the isotropic material is directional independent. Application of the normal stress causes extension in the direction of the applied stress and contraction in the perpendicular direction. In addition, shear stress causes only shearing deformation.

The deformation of the anisotropic is directional dependent. Application of the normal stress leads not only to extension in the direction of the stress and contraction perpendicular to it, but to shearing deformation as well. Conversely, shearing stress causes extension and contraction in addition to the distortion of shearing deformation.

The deformation of the orthotropic material subjected to the normal stress in the principal material direction is similar to one of the isotropic material. However, due to the different properties in the two principal directions, the contraction can be either more or less than the contraction of a similarly loaded isotropic material with the same modulus of elasticity in the direction of load. In addition, shearing stress causes shearing deformation, but the magnitude of the deformation is independent of the various modulus of elasticity and Poisson's ratios.

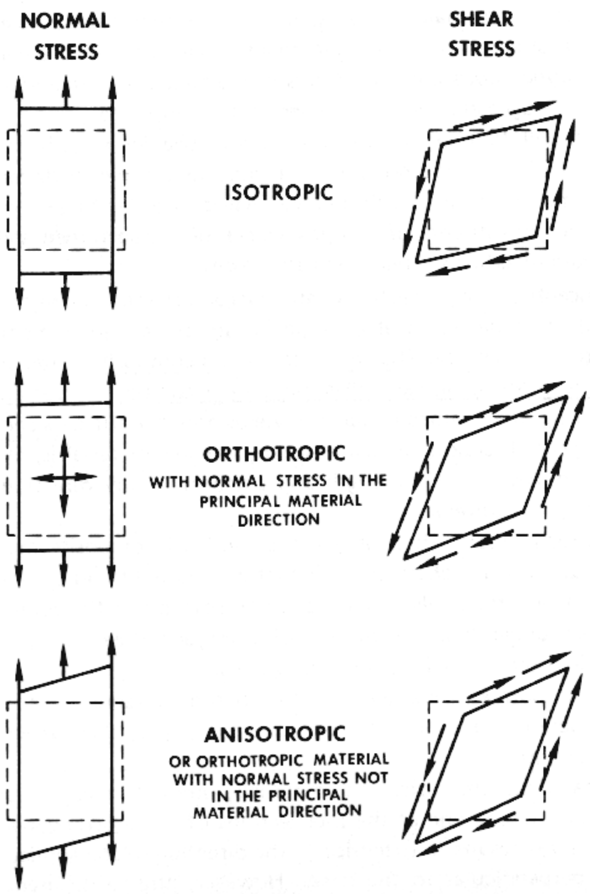


Fig. 2.9

The followings are the strain-stress relations corresponding to the stress-strain relations of the anisotropic material, monoclinic material, orthotropic material, transversely isotropic material, and isotropic material. Note that the term S_{ij}, $i, j = 1, 2, 3, 4, 5, 6$ is called the compliance coefficients.

Anisotropic material

The strain-stress relations of the anisotropic material have 21 independent compliance coefficients

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Monoclinic material

The strain-stress relations of the monoclinic material have 13 independent compliance coefficients. For symmetry about the axis $z = 0$,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Orthotropic material

The strain-stress relations of the orthotropic material have 9 independent compliance coefficients.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Transversely isotropic material

The strain-stress relations of the transversely isotropic material have 5 independent compliance coefficients. For symmetry about the plane $x - y$,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (S_{11} - S_{12})/2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Isotropic material

The strain-stress relations of the isotropic material have 2 independent compliance coefficients.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & (S_{11} - S_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (S_{11} - S_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (S_{11} - S_{12})/2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

2.4 Strain Energy of an Infinitesimal Small Element

Anisotropic materials

Consider the state of stresses of an infinitesimal small element cut from a body subjected to external loads as shown in Fig. 2.10.

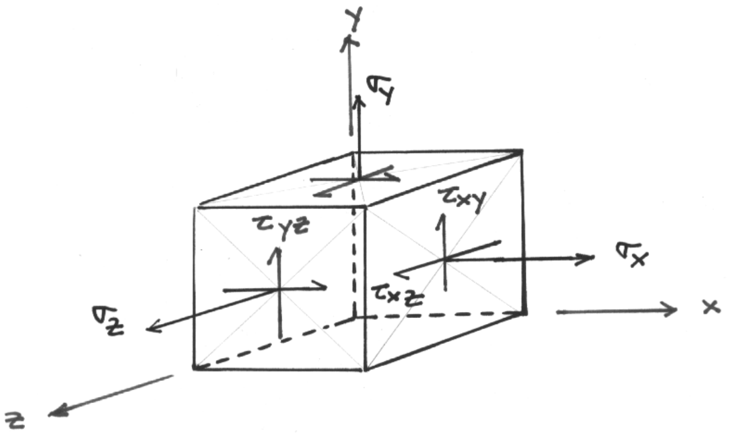


Fig. 2.10

By integrating the strain energy density equation ($dU_o = \sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \sigma_z d\varepsilon_z + \tau_{xy} d\gamma_{xy} + \tau_{yz} d\gamma_{yz} + \tau_{xz} d\gamma_{xz}$), from the initial to the final state of strain of a body, the strain energy density of the anisotropic material will be obtained.

Let k is **a constant** used to indicate the state of strain of a body. It has a value from 0 at the initial state to 1 at the final state. At a given point, the stress components are $k\sigma_x$, $k\sigma_y$, $k\sigma_z$, $k\tau_{xy}$, $k\tau_{yz}$, and $k\tau_{xz}$. If the incremental strain components at that point are $(dk)\varepsilon_x$, $(dk)\varepsilon_y$, $(dk)\varepsilon_z$, $(dk)\gamma_{xy}$, $(dk)\gamma_{yz}$, and $(dk)\gamma_{xz}$. Hence,

$$\int_0^1 U_o(dk) = (C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\varepsilon_z + C_{14}\gamma_{yz} + C_{15}\gamma_{xz} + C_{16}\gamma_{xy})\,\varepsilon_x \int_0^1 kdk +$$

$$(C_{12}\varepsilon_x + C_{22}\varepsilon_y + C_{23}\varepsilon_z + C_{24}\gamma_{yz} + C_{25}\gamma_{xz} + C_{26}\gamma_{xy})\,\varepsilon_y \int_0^1 kdk +$$

$$(C_{13}\varepsilon_x + C_{23}\varepsilon_y + C_{33}\varepsilon_z + C_{34}\gamma_{yz} + C_{35}\gamma_{xz} + C_{36}\gamma_{xy})\,\varepsilon_z \int_0^1 kdk +$$

$$(C_{14}\varepsilon_x + C_{24}\varepsilon_y + C_{34}\varepsilon_z + C_{44}\gamma_{yz} + C_{45}\gamma_{xz} + C_{46}\gamma_{xy})\,\gamma_{yz} \int_0^1 kdk +$$

$$(C_{15}\varepsilon_x + C_{25}\varepsilon_y + C_{35}\varepsilon_z + C_{45}\gamma_{yz} + C_{55}\gamma_{xz} + C_{56}\gamma_{xy})\,\gamma_{xz} \int_0^1 kdk +$$

$$(C_{16}\varepsilon_x + C_{26}\varepsilon_y + C_{36}\varepsilon_z + C_{46}\gamma_{yz} + C_{56}\gamma_{xz} + C_{66}\gamma_{xy})\,\gamma_{xy} \int_0^1 kdk$$

$$U_o = \frac{1}{2}(C_{11}\varepsilon_x^2 + C_{12}\varepsilon_x\varepsilon_y + C_{13}\varepsilon_x\varepsilon_z + C_{14}\varepsilon_x\gamma_{yz} + C_{15}\varepsilon_x\gamma_{xz} + C_{16}\varepsilon_x\gamma_{xy}) +$$

$$\frac{1}{2}(C_{12}\varepsilon_y\varepsilon_x + C_{22}\varepsilon_y^2 + C_{23}\varepsilon_y\varepsilon_z + C_{24}\varepsilon_y\gamma_{yz} + C_{25}\varepsilon_y\gamma_{xz} + C_{26}\varepsilon_y\gamma_{xy}) +$$

$$\frac{1}{2}(C_{13}\varepsilon_z\varepsilon_x + C_{23}\varepsilon_z\varepsilon_y + C_{33}\varepsilon_z^2 + C_{34}\varepsilon_z\gamma_{yz} + C_{35}\varepsilon_z\gamma_{xz} + C_{36}\varepsilon_z\gamma_{xy}) +$$

$$\frac{1}{2}(C_{14}\gamma_{yz}\varepsilon_x + C_{24}\gamma_{yz}\varepsilon_y + C_{34}\gamma_{yz}\varepsilon_z + C_{44}\gamma_{yz}^2 + C_{45}\gamma_{yz}\gamma_{xz} + C_{46}\gamma_{yz}\gamma_{xy}) +$$

$$\frac{1}{2}(C_{15}\gamma_{xz}\varepsilon_x + C_{25}\gamma_{xz}\varepsilon_y + C_{35}\gamma_{xz}\varepsilon_z + C_{45\gamma_{xz}}\gamma_{yz} + C_{55}\gamma_{xz}^2 + C_{56}\gamma_{xz}\gamma_{xy}) +$$

$$\frac{1}{2}(C_{16}\gamma_{xy}\varepsilon_x + C_{26}\gamma_{xy}\varepsilon_y + C_{36}\gamma_{xy}\varepsilon_z + C_{46}\gamma_{xy}\gamma_{yz} + C_{56}\gamma_{xy}\gamma_{xz} + C_{66}\gamma_{xy}^2)$$

In the index notation, we have

$$U_o = \frac{1}{2}C_{ij}\varepsilon_i\varepsilon_j$$

Isotropic and homogenous materials

As previously mentioned, a material is isotropic when the mechanical properties of the material are symmetric on an infinite number of plane. In other words, its mechanical properties are invariant under any rotation of coordinates. A material is *homogenous* when the mechanical properties of the material are identical for every point in a body. In other words, its mechanical properties are invariant under any translation of coordinates. Thus, if the material of an elastic body is isotropic, the strain energy density depends only on the principal strains ε_1, ε_2, and ε_3, which are invariant under arbitrary rotation.

$$U_o = \frac{1}{2}C_{11}\varepsilon_1^2 + \frac{1}{2}C_{12}\varepsilon_1\varepsilon_2 + \frac{1}{2}C_{13}\varepsilon_1\varepsilon_3 +$$

$$\frac{1}{2}C_{12}\varepsilon_1\varepsilon_2 + \frac{1}{2}C_{22}\varepsilon_2^2 + \frac{1}{2}C_{23}\varepsilon_2\varepsilon_3 +$$

$$\frac{1}{2}C_{13}\varepsilon_1\varepsilon_3 + \frac{1}{2}C_{23}\varepsilon_2\varepsilon_3 + \frac{1}{2}C_3\varepsilon_3^2$$

Since the mechanical properties of the isotropic material are symmetric for all planes, the naming of the principal axes is arbitrary. Thus, the isotropic material has only two distinct coefficients.

$$C_{11} = C_{22} = C_{33} = C_1$$

$$C_{12} = C_{23} = C_{13} = C_2$$

$$U_o = \frac{1}{2}C_1\varepsilon_1^2 + \frac{1}{2}C_1\varepsilon_2^2 + \frac{1}{2}C_1\varepsilon_3^2 + C_2\varepsilon_1\varepsilon_2 + C_2\varepsilon_1\varepsilon_3 + C_2\varepsilon_2\varepsilon_3$$

$$U_o = \frac{C_1}{2}(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) + C_2(\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3)$$

Rewriting the term $C_2(\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3)$ and noting that

$$(a+b+c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$$

$$(ab + ac + bc) = \frac{1}{2}\left[(a+b+c)^2 - (a^2 + b^2 + c^2)\right]$$

Hence,

$$C_2(\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3) = \frac{C_2}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 - \frac{C_2}{2}(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)$$

Then, the strain energy density can be rewritten as

$$U_o = \frac{C_1}{2}(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) + \frac{C_2}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 - \frac{C_2}{2}(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)$$

Let $\lambda = C_2$ and $G = \dfrac{C_1 - C_2}{2}$ are elastic constants called Lame's elastic coefficients. Thus,

$$U_o = \frac{1}{2}\lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 + G(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)$$

In terms of the strain invariants, $J_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ and $J_2 = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3$. Rewriting the second term of the strain energy density equation, we have

$$G(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) = G(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 - 2G(\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3)$$

Then,

$$U_o = \frac{1}{2}\lambda J_1^2 + GJ_1^2 - 2GJ_2$$

In general, $J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$ and $J_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2)$, then,

$$U_o = \frac{1}{2}\lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + G\ (\varepsilon_x + \varepsilon_y + \varepsilon_z)^2$$

$$- 2G\ (\varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2))$$

$$U_o = \frac{1}{2}\lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + G(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + 2G(\varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_z)$$

$$- 2G\ (\varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2))$$

$$U_o = \frac{1}{2}\lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + G(\ \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \frac{1}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2))$$

Since $\sigma_x = \frac{\partial U_o}{\partial \varepsilon_x}$, $\sigma_y = \frac{\partial U_o}{\partial \varepsilon_y}$, $\sigma_z = \frac{\partial U_o}{\partial \varepsilon_z}$, $\tau_{xy} = \frac{1}{2}\frac{\partial U_o}{\partial \varepsilon_{xy}} = \frac{\partial U_o}{\partial \gamma_{xy}}$, $\tau_{yz} = \frac{1}{2}\frac{\partial U_o}{\partial \varepsilon_{yz}} = \frac{\partial U_o}{\partial \gamma_{yz}}$, $\tau_{xz} = \frac{1}{2}\frac{\partial U_o}{\partial \varepsilon_{xz}} = \frac{\partial U_o}{\partial \gamma_{xz}}$, consequently,

$$\sigma_x = \frac{\partial}{\partial \varepsilon_x}[\frac{1}{2}\lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + G(\ \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \frac{1}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2))]$$

$$\sigma_x = \lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_x = \lambda e + 2G\varepsilon_x$$

$$\sigma_y = \frac{\partial}{\partial \varepsilon_y}[\frac{1}{2}\lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + G(\ \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \frac{1}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2))]$$

$$\sigma_y = \lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_y = \lambda e + 2G\varepsilon_y$$

$$\sigma_z = \frac{\partial}{\partial \varepsilon_z}[\frac{1}{2}\lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + G(\ \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \frac{1}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2))]$$

$$\sigma_z = \lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_z = \lambda e + 2G\varepsilon_z$$

$$\tau_{yz} = \frac{\partial}{\partial \gamma_{yz}}[\frac{1}{2}\lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + G(\ \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \frac{1}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2))]$$

$$\tau_{yz} = G\gamma_{yz}$$

$$\tau_{xz} = \frac{\partial}{\partial \gamma_{xz}}[\frac{1}{2}\lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + G(\ \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \frac{1}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2))]$$

$$\tau_{xz} = G\gamma_{xz}$$

$$\tau_{xy} = \frac{\partial}{\partial \gamma_{xy}}[\frac{1}{2}\lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + G(\ \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \frac{1}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2))]$$

$$\tau_{xy} = G\gamma_{xy}$$

where $e \cong J_1$ is the classical small-displacement cubical strain. The stress invariants can be related with the strain invariants by

$$I_1 = \sigma_x + \sigma_y + \sigma_z = \lambda e + 2G\varepsilon_x + \lambda e + 2G\varepsilon_y + \lambda e + 2G\varepsilon_z$$

$$I_1 = 3\lambda e + 2G(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$I_1 = (3\lambda + 2G)J_1$$

$$I_2 = \lambda(3\lambda + 4G)J_1^2 + 4G^2 J_2$$

$$I_3 = \lambda^2(\lambda + 2G)J_1^3 + 4\lambda G^2 J_1 J_2 + 8G^3 J_3$$

Inverting the stress equations, we obtain

$$\gamma_{yz} = 2\varepsilon_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{xz} = 2\varepsilon_{xz} = \frac{\tau_{xz}}{G}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \frac{\tau_{xy}}{G}$$

$$\sigma_x = \lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_x = (\lambda + 2G)\varepsilon_x + \lambda\varepsilon_y + \lambda\varepsilon_z$$

$$\sigma_y = \lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_y = \lambda\varepsilon_x + (\lambda + 2G)\varepsilon_y + \lambda\varepsilon_z$$

$$\sigma_z = \lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_z = \lambda\varepsilon_x + \lambda\varepsilon_y + (\lambda + 2G)\varepsilon_z$$

In matrix notation, we have

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} = \begin{bmatrix} (\lambda+2G) & \lambda & \lambda \\ \lambda & (\lambda+2G) & \lambda \\ \lambda & \lambda & (\lambda+2G) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} = \begin{bmatrix} (\lambda+2G) & \lambda & \lambda \\ \lambda & (\lambda+2G) & \lambda \\ \lambda & \lambda & (\lambda+2G) \end{bmatrix}^{-1} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix}$$

This matrix inversion can be performed by hand or by computer program such as Mathematica, and we obtain.

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

where $E = \dfrac{G(3\lambda + 2G)}{\lambda + G}$, $\nu = \dfrac{\lambda}{2(\lambda + G)}$, and $G = \dfrac{E}{2(1+\nu)}$. In matrix notation,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

The strain energy density of isotropic material can be found by substituting the strains into the equation $U_o = \frac{1}{2}\lambda\ (\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + G(\ \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \frac{1}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2))$ and rearranging the terms.

$$U_o = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{2\nu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2)$$

or,

$$U_o = \frac{1}{2}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}\right]$$

2.5 Stress-Strain Relations for Isotropic Material: Physical Derivation

Consider a cubic volume element subjected to a state of triaxial normal stress σ_x, σ_y, σ_z and associated normal strains ε_x, ε_y, and ε_z are developed in the material. Since the material is isotropic, the cubic volume element will deform to a rectangular block, no shear strains is produced in the material. By using the principle of superposition, the deformation of the cubic volume element subjected to each normal stress can be draw as shown in Fig. 2.11.

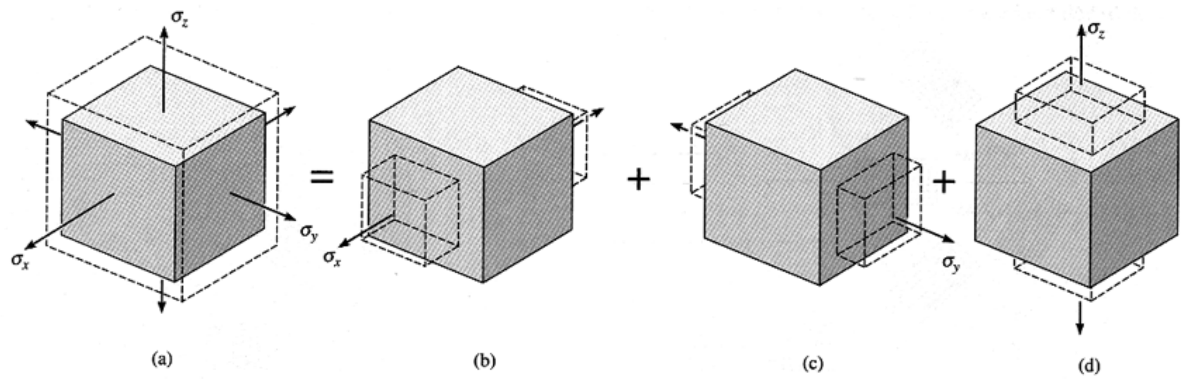


Fig. 2.11

First, consider the normal strain of the element in the x direction, caused by separate application of each normal stress σ_x, σ_y, and σ_z. Under σ_x, the cubic volume element elongates in the x direction and the associated strain in this direction is

$$\varepsilon_x' = \frac{\sigma_x}{E}$$

When σ_y is applied, the cubic volume element contracts in the x direction due to the Poisson's effects and the associated strain in this direction is

$$\varepsilon_x'' = -\nu\frac{\sigma_y}{E}$$

Similarly, when σ_z is applied, the cubic volume element contracts in the x direction due to the Poisson's effects and the associated strain in this direction is

$$\varepsilon_x''' = -\nu \frac{\sigma_z}{E}$$

Superimposing these three normal strains, we have the total normal stress σ_x of the state of stress equals to

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

Likewise, the normal strain in the y and z direction can be determined as

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

In the test, application of the shear stress τ_{xy} to the cubic volume element of the isotropic material only produces the shear strain γ_{xy} in the element as shown in Fig. 2.12a. Likewise, the shear stresses τ_{yz} and τ_{zx} only produce the shear strain γ_{yz} and γ_{xz} on the cubic volume element. Hence,

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz}$$

$$\gamma_{zx} = \frac{1}{G}\tau_{zx}$$

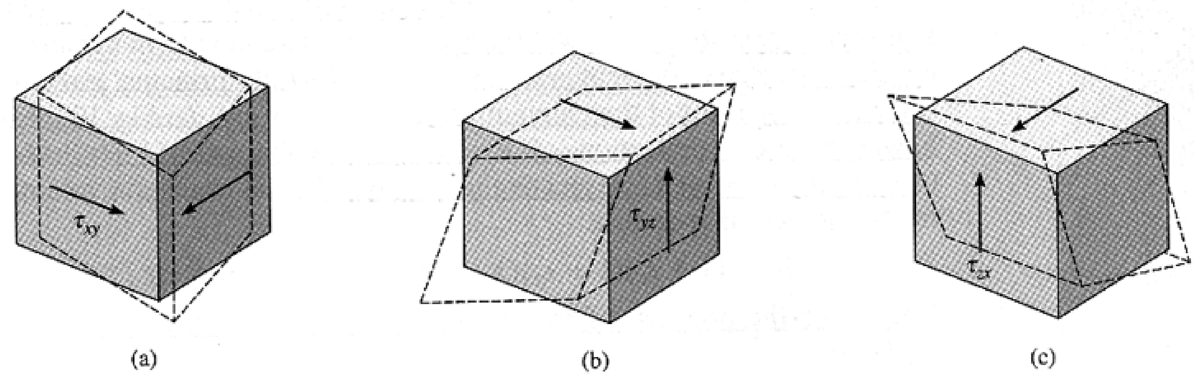


Fig. 2.12

Example 2-2

The thin-walled cylindrical pressure vessel 10 m long as shown in Fig. Ex2-2 has closed ends, a wall thickness of 5 mm, and an inner diameter of 3 m. If the vessel is filled with air to a pressure of $2\,\mathrm{MPa}$, how much do the length, diameter, and wall thickness change, and in each case is the change an increase or a decrease? The vessel is made of steel having $E = 200\,\mathrm{GPa}$, and $\nu = 0.30$.

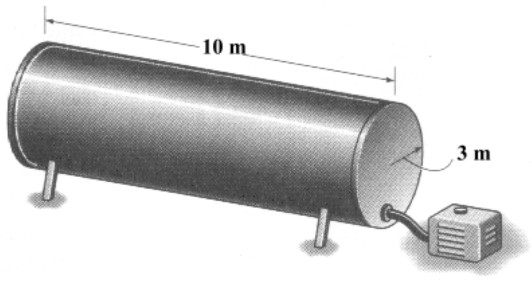


Fig. Ex2-2

Let the x-axis is along the longitudinal axis of the vessel and the z-axis is normal to the surface. Thus, the y-axis is in the tangential direction.

Since the ratio of the radius to the thickness, r/t, is small, thus,

$$\sigma_x = \frac{pr}{2t} = \frac{2(1.5)}{2(0.005)} = 300\,\mathrm{MPa}$$

$$\sigma_y = \frac{pr}{t} = \frac{2(1.5)}{(0.005)} = 600\,\mathrm{MPa}$$

The value of σ_z varies from $-p$ on the inside wall to zero on the outside wall, thus, $\sigma_z \approx 0$ and we have

$$\varepsilon_x = \frac{1}{200(10^3)}\left[300 - 0.3(600+0)\right] = 0.00060$$

$$\varepsilon_y = \frac{1}{200(10^3)}\left[600 - 0.3(300+0)\right] = 0.00255$$

$$\varepsilon_z = \frac{1}{200(10^3)}\left[0 - 0.3(300+600)\right] = -0.00135$$

Since $\varepsilon_x = \dfrac{\Delta L}{L}$, $\varepsilon_y = \dfrac{\Delta(\pi d)}{\pi d} = \dfrac{\Delta d}{d}$, and $\varepsilon_z = \dfrac{\Delta t}{t}$,

$$\Delta L = 0.00060(10)10^3 = +6\,\mathrm{mm}$$

$$\Delta d = 0.00255(3)10^3 = +7.65\,\mathrm{mm}$$

$$\Delta t = -0.00135(5) = -6.75(10^{-3})\,\mathrm{mm}$$

Thus, there are small increases in length and diameter, and a tiny decrease in the wall thickness.

Example 2-3

A sample of material subjected to a compressive stress σ_z is confined so that it can not deformed in the y-direction, but deformation is permitted in the x-direction, as shown in Fig. Ex 2-3. Assuming that the material is isotropic and exhibits linear-elastic behavior. Determine the following in term of σ_z and the elastic constants of the material:

a.) The stress that develops in the y-direction.

b.) The strain in the z-direction.

c.) The strain in the x-direction.

d.) The stiffness $E' = \sigma_z / \varepsilon_z$ in the z-direction. Is this apparent modulus equal to the elastic modulus E from the uniaxial test on the material? Why or why not?

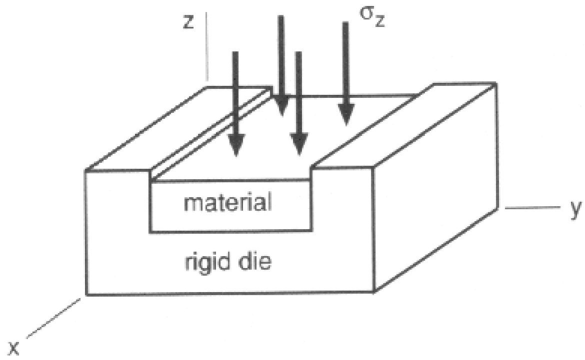


Fig. Ex 2-3

Since the sample can not deformed in the y-direction, $\varepsilon_y = 0$, and since the deformation is permitted in the x-direction, $\sigma_x = 0$.

The stress that develops in the y-direction is

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$0 = \frac{1}{E}\left[\sigma_y - \nu(0 + \sigma_z)\right]$$

$$\sigma_y = \nu\sigma_z$$

The strain in the z-direction is

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(0 + \nu\sigma_z)\right]$$

$$\varepsilon_z = \frac{1-\nu^2}{E}\sigma_z$$

The strain in the x-direction is

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_x = \frac{1}{E}\left[0 - \nu(\nu\sigma_z + \sigma_z)\right]$$

$$\varepsilon_x = -\frac{\nu(1+\nu)}{E}\sigma_z$$

The stiffness $E' = \sigma_z / \varepsilon_z$ is

$$E' = \frac{E}{1-\nu^2}$$

Thus, the apparent stiffness differs from the elastic modulus E from the uniaxial test. This is due to the fact that the apparent stiffness is determined by behavior according to the three-dimensional form of Hooke's law.

Chapter 3

Elements of Theory of Elasticity

3.1 Introduction

In the analysis of a body or structure, the geometry of the structure and the loads are given. A solution may be obtained by analytical, numerical, and experimental methods. In the analytical methods, the derivation of the load-stress relations depends on the following conditions:

1. The equilibrium equations.
2. The compatibility equations.
3. The stress-strain relations.
4. The material responses.

Two different analytical methods used to satisfy the first and second condition are the method of mechanics of materials and the theory of elasticity method.

The mechanics of materials involve the following steps.

1. The simplified assumptions related to the geometry of the deformation of the structure are established by using the compatibility equations.
2. Analyze the geometry of the deformation to determine the strain distributions over a cross section of the structure.
3. Relate the applied loads to the internal stress by using the equilibrium equations.
4. Use the stress-strain relations and the material responses to determine the relations between the assumed strain distribution and stress distribution over a cross section of the structure.
5. Relate the applied loads to the displacement of the structure.

The obtained results may be exact, or good approximations, or rough estimate, depending largely on the accuracy of the assumptions made in the first step.

In the theory of elasticity, the states of stresses and displacements for every point in the structure are determined by simultaneously satisfy the requirements of equilibrium at every point, compatibility of all displacements and boundary conditions on stress and displacement. This method involves no initial assumptions or approximation about the geometry of the deformation. Thus, the method is more difficult than the mechanics of materials. However, it is usually used to solve the problems in which the geometry of the deformation can not be reliably anticipated such as determining the stress concentration occurred at a hole in a plate.

Often, a practical problem is solves by using both methods simultaneously.

Limitations

In this chapter, we are considering that the material is homogeneous, isotropic, and linearly elastic, and that the displacements are small.

Example 3-1: One dimensional problem

Determine the states of stresses and the axial displacement for every point in the bar shown in Fig. Ex 3-1a.

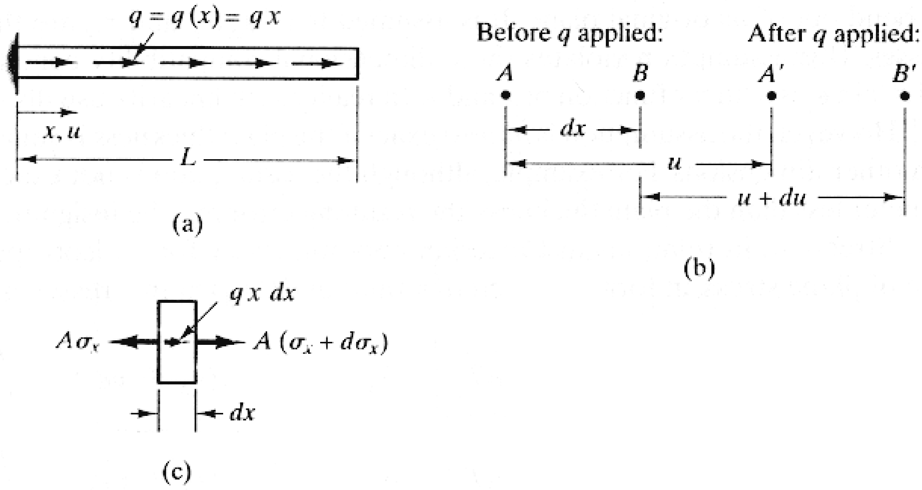


Fig. Ex 3-1

To determine the states of stresses for every point in the bar shown in Fig. Ex 3-1a, we are *assuming that the bar is in a state of uniaxial stress*. This idealization is not exact since the state of stress at the fixed end of the bar is in three dimensions due to the Poisson's effects. However, according to Saint-Venant's principle, the effects are localized.

1. By using the equilibrium of forces in the longitudinal axis of the differential element shown in Fig. Ex 3-1c, we have

$$-A\sigma_x + A(\sigma_x + d\sigma_x) + q(x)dx = 0$$

$$\frac{d\sigma_x}{dx} + \frac{q(x)}{A} = 0$$

This equation is a differential equation of equilibrium, which must be satisfied for every point in the bar from $x = 0$ to $x = L$. Let $q = qx$ and the bar is prismatic, the states of stresses can be determined by integrating the previously obtained equation.

$$\sigma_x = \frac{1}{A}\int_0^x qx \quad dx = \frac{1}{A}\frac{qx^2}{2} + C_1$$

The integration constant C_1 can be found by using the boundary condition of stress: at $x = 0$, $\sigma_x = 0$. Hence, $C_1 = -\frac{qL^2}{2A}$ and

$$\sigma_x = \frac{qx^2}{2A} - \frac{qL^2}{2A} = \frac{q}{2A}(x^2 - L^2)$$

2. To determine the axial displacement, we need first consider how the differential element deformed under the load $q = qx$ and, then, obtain the strain-displacement relation.

$$\varepsilon_x = \frac{(u+du)-u}{dx} = \frac{du}{dx}$$

The uniaxial stress-strain relation, $\sigma_x = E\varepsilon_x$.

$$\sigma_x = E\frac{du}{dx}$$

Substituting σ_x into the differential equation of equilibrium, we have

$$E\frac{d^2u}{dx^2} + \frac{qx}{A} = 0$$

Performing the first integration with respect to x and using the displacement boundary condition: at $x = L$, $\frac{du}{dx} = 0$, we obtain

$$E\frac{du}{dx} = -\frac{1}{A}\int_0^x qx \quad dx = -\frac{qx^2}{2A} + C_2$$

$$C_2 = \frac{qL^2}{2A}$$

$$E\frac{du}{dx} == \frac{q}{2A}(L^2 - x^2)$$

Performing the second integration with respect to x and using the displacement boundary condition: at $x = 0$, $u = 0$, we obtain

$$Eu = \frac{q}{2A}\int_0^x (L^2 - x^2)dx = \frac{q}{2A}\left[L^2 x - \frac{x^3}{3}\right] + C_3$$

$$C_3 = 0$$

Therefore, the displacement equation is

$$u = \frac{qx}{2AE}\left[L^2 - \frac{x^2}{3}\right]$$

3.2 Two-Dimensional Problems of Theory of Elasticity

Consider a plate structure subjected to the external load parallel to the plate as shown in Fig. 3-1. If the plate is very thin compared to the dimension of the plate, we can proved that the stresses σ_z, τ_{xz}, and τ_{yx} on an infinitesimal small element far away from the loading points are approximately equal to zero. This kind of state of stresses on the infinitesimal small element is called the *plane stress*. The strain-stress relations of the plane stress are

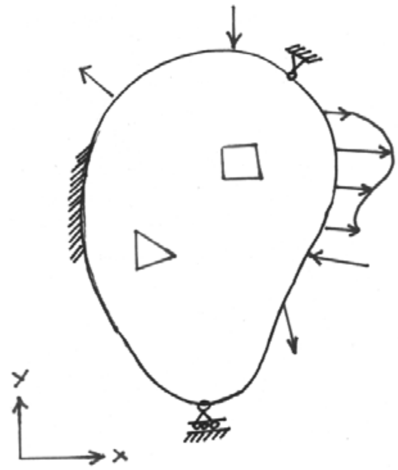


Fig. 3.1

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ 0 \\ 0 \\ 0 \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$G = \frac{E}{2(1+\nu)}$$

and the stress-strain relations of the plane stress are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

The strain-displacement relations of the plane stress are

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

The strain compatibility relation of the plane stress is

$$\frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

The differential equations of equilibrium of the plane stress are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + B_y = 0$$

Example 3-2

Given that the body force is negligible. Investigate if the following displacement field can be a solution of a static plane stress problem

$$u = a_1(x^2 - y^2) - a_2 y + a_3 \text{ and } v = 2a_1 xy + a_4$$

where the a_i are constants.

The strain-displacement relations are

$$\varepsilon_x = \frac{\partial u}{\partial x} = 2a_1 x \qquad \varepsilon_y = \frac{\partial v}{\partial y} = 2a_1 x$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = (-2a_1 y - a_2) + 2a_1 y = -a_2$$

The strain compatibility relation is satisfied since

$$\frac{\partial^2 \varepsilon_y}{\partial x^2} = 0 \qquad \frac{\partial^2 \varepsilon_x}{\partial y^2} = 0 \qquad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0$$

The stress-strain relations of the plane stress are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} 2a_1 x \\ 2a_1 x \\ -a_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} \dfrac{2Ea_1 x}{1-\nu} \\ \dfrac{2Ea_1 x}{1-\nu} \\ -\dfrac{a_2 E}{2(1+\nu)} \end{Bmatrix}$$

The differential equations of equilibrium of the plane stress are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \neq 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

Thus, the displacement field is not a possible solution.

Boundary condition

Boundary condition is the prescribed condition of displacements and forces at the boundaries of a structure. Consider a body as shown in Fig. 3.2.

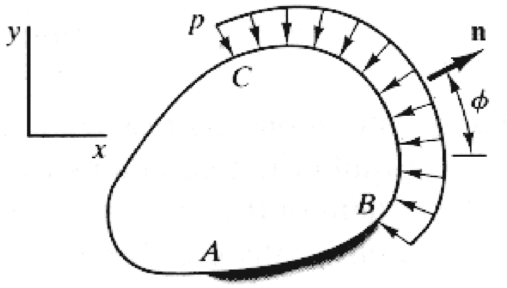


Fig. 3.2

1. Geometric or displacement boundary condition pertains to the compatibility conditions and required that the displacement (including rotation) at the structural boundary must be satisfied.

Along the boundary AB which is the fixed support, the displacements u and v are zero.

2. Natural or force boundary condition pertains to the equilibrium conditions and required that the forces (including moment) at the structural boundary must be satisfied.

From chapter 1, the stress vector $\vec{\sigma}_P$ on an arbitrary plane in three-dimension is

$$\vec{\sigma}_P = \sigma_{Px}\hat{i} + \sigma_{Py}\hat{j} + \sigma_{Pz}\hat{k}$$

where

$$\sigma_{Px} = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$

$$\sigma_{Py} = \tau_{xy} l + \sigma_y m + \tau_{yz} n$$

$$\sigma_{Pz} = \tau_{xz} l + \tau_{yz} m + \sigma_z n$$

From Fig. 3.2, the direction cosine along the boundary BC and AC is $l = \cos\phi$, $m = \sin\phi$, and $n = 0$. Thus,

$$\sigma_{Px} = \sigma_x \cos\phi + \tau_{xy} \sin\phi$$

$$\sigma_{Py} = \tau_{xy} \cos\phi + \sigma_y \sin\phi$$

Defining *surface traction* as $\Phi_x = \dfrac{dF_x}{dA}$, $\Phi_y = \dfrac{dF_y}{dA}$, and $\Phi_z = \dfrac{dF_y}{dA}$ where F_x, F_y, and F_z are the components of a force vector $\vec{F}$ acting on the surface having a boundary area dA. Hence,

$$\Phi_x = -\frac{(pdA)\cos\phi}{dA} = -p\cos\phi$$

$$\Phi_y = \frac{dF_y}{dA} = -\frac{(pdA)\sin\phi}{dA} = -p\sin\phi$$

The force boundary conditions along the boundary BC are

$\sigma_{Px} = \Phi_x$; $\quad -p\cos\phi = \sigma_x \cos\phi + \tau_{xy}\sin\phi$

$\sigma_{Py} = \Phi_y$; $\quad -p\sin\phi = \tau_{xy}\cos\phi + \sigma_y \sin\phi$

At $\phi = 0$, $\quad \sigma_x = -p$

$\tau_{xy} = 0$

At $\phi = \frac{\pi}{2}$, $\quad \sigma_y = -p$

$\tau_{xy} = 0$

The force boundary conditions along the boundary BC are

At $\phi = \pi$, $\quad \sigma_x = 0$

$\tau_{xy} = 0$

$\sigma_y = 0$

The elasticity solutions of this example must satisfy the above displacement and force boundary conditions.

Example 3-3

The cantilevered beam as shown in Fig. Ex 3-3 has unit thickness. Neglecting the body forces, determine

a.) The flexural stress σ_x due to the applied load by using the mechanics of material method of analysis.

b.) The stresses σ_y and γ_{xy} by using the obtained the flexural stress σ_x.

c.) If the obtained state of stresses is possible for the theory of elasticity method of analysis?

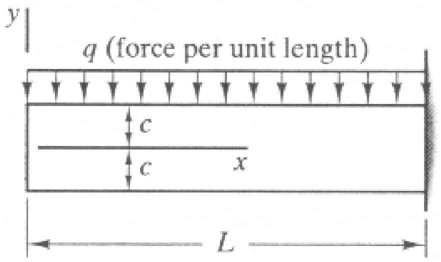


Fig. Ex 3-3

By the mechanics of material method of analysis, we have

$$\sigma_x = \frac{My}{I} = \frac{\left(\frac{qx^2}{2}\right)y}{\left(\frac{2c^3}{3}\right)} = \frac{3q}{4c^3}x^2 y$$

To find τ_{xy}, we use the differential equation of equilibrium of the plane stress where the body forces is neglected. Thus,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{3q}{2c^3}xy + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\tau_{xy} = -\int \frac{3q}{2c^3}xydy + C_1$$

$$\tau_{xy} = -\frac{3q}{4c^3}xy^2 + C_1$$

Stress boundary condition, at $y = +c$, $\tau_{xy} = 0$.

$$C_1 = \frac{3q}{4c^3}xc^2$$

$$\tau_{xy} = \frac{3q}{4c^3}x(c^2 - y^2)$$

To find σ_y, we use the differential equation of equilibrium of the plane stress where the body forces is neglected. Thus,

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0$$

$$\frac{3q}{4c^3}(c^2-y^2)+\frac{\partial \sigma_y}{\partial y}=0$$

$$\sigma_y=-\int \frac{3q}{4c^3}(c^2-y^2)dy+C_2$$

$$\sigma_y=-\frac{3q}{4c^3}(c^2y-\frac{y^3}{3})+C_2$$

Stress boundary condition, at $y=+c$, $\sigma_y=-q$.

$$-q=-\frac{3q}{4c^3}(c^3-\frac{c^3}{3})+C_2$$

$$C_2=-\frac{q}{2}$$

$$\sigma_y=\frac{q}{2}\left[\frac{y^3}{2c^3}-\frac{3y}{2c}-1\right]$$

To determine if the obtained state of stresses is possible, we use the strain compatibility equation,

$$\frac{\partial^2 \varepsilon_y}{\partial x^2}+\frac{\partial^2 \varepsilon_x}{\partial y^2}=\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

But, we need to find the strains from the stresses first. For the plane stress problem,

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{Bmatrix}\frac{3q}{4c^3}x^2y\\ \frac{q}{2}\left[\frac{y^3}{2c^3}-\frac{3y}{2c}-1\right]\\ \frac{3q}{4c^3}x(c^2-y^2)\end{Bmatrix}$$

Then, substituting the obtained strains into the strain compatibility equation, we get

$$\frac{\partial^2 \varepsilon_y}{\partial x^2}+\frac{\partial^2 \varepsilon_x}{\partial y^2}\neq\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Therefore, the state of stresses is impossible for the theory of elasticity method of analysis.

3.3 Stress Field Solution for Plane Stress Problem

To solve problems in either plane stress and plane strain, one may begin by finding stresses that satisfy the equation of equilibrium.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + B_y = 0$$

The above two equations have three unknowns stresses. Hence, an infinite number of stress solutions can be obtained. Let the body forces per unit volume are defined as derivatives of a *potential function* $V = V(x, y)$ where

$$B_x = -\frac{\partial V}{\partial x} \qquad\qquad B_y = -\frac{\partial V}{\partial y}$$

In addition, let $F = F(x, y)$ be an arbitrary function and let

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} + V \qquad\qquad \sigma_y = \frac{\partial^2 F}{\partial x^2} + V \qquad\qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

These functions are satisfied the equations of equilibrium.

$$\left[\frac{\partial^3 F}{\partial x \partial y^2} + \frac{\partial V}{\partial x}\right] - \frac{\partial^3 F}{\partial x \partial y^2} - \frac{\partial V}{\partial x} = 0$$

$$-\frac{\partial^3 F}{\partial x^2 \partial y} + \left[\frac{\partial^3 F}{\partial x^2 \partial y} + \frac{\partial V}{\partial y}\right] - \frac{\partial V}{\partial y} = 0$$

Therefore, as far as the equations of equilibrium are concerned, the assumed stress functions constitute a general solution. The problem is reduced to finding a function $F = F(x, y)$ that satisfies the compatibility conditions and the boundary conditions.

The strain-stress relations of the plane stress problems are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 F}{\partial y^2} + V \\ \dfrac{\partial^2 F}{\partial x^2} + V \\ -\dfrac{\partial^2 F}{\partial x \partial y} \end{Bmatrix}$$

The strain compatibility relation of the plane stress is $\dfrac{\partial^2 \varepsilon_y}{\partial x^2} + \dfrac{\partial^2 \varepsilon_x}{\partial y^2} = \dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y}$.

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{1}{E}\left[\frac{\partial^4 F}{\partial y^4} - \nu \frac{\partial^4 F}{\partial x^2 \partial y^2} + (1-\nu)\frac{\partial^2 V}{\partial y^2}\right]$$

$$\frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{1}{E}\left[-\nu \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial x^4} + (1-\nu)\frac{\partial^2 V}{\partial x^2}\right]$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = -\frac{2(1+\nu)}{E}\frac{\partial^4 F}{\partial x^2 \partial y^2}$$

The only compatibility equation not identically satisfied is

$$\left[\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4}\right] + (1-\nu)\left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right] = 0$$

$$\nabla^4 F + \nabla^2 V = 0$$

Now, the problem is reduced to the solution of boundary-value problem associated with *biharmonic differential operator* $\nabla^4 F$ and *harmonic differential operator* $\nabla^2 V$ which can be solved by using theory of functions of a complex variable.

If the body forces are absent,

$$\nabla^4 F = 0$$

The function $F = F(x, y)$ is called the *Airy stress function* which is discovered by Sir George Biddell Airy in 1863.

Example 3-4

A rectangular block having the mass density ρ stands on a rigid horizontal support and is loaded by its own weight as shown in Fig. Ex 3-4a. As a proposed solution of this problem, investigate the equation

$$F = -\rho g\left(\frac{y^3}{6} + \frac{Lx^2}{2}\right) \text{ and } V = \rho g y$$

Determine the displacement $u = u(x, y)$ and $v = v(x, y)$ of the block due to its own weight.

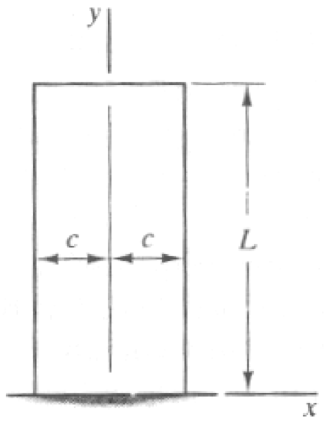


Fig. Ex 3-4a

From the Airy stress function, we obtain

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} + V = -\rho g y + \rho g y = 0$$

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} + V = -\rho g L + \rho g y = \rho g (y - L)$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = 0$$

The results satisfy the free-surface boundary conditions of the top and the vertical edges of the block where, for the top edge,

$$\sigma_x = 0$$

$$\sigma_y = \rho g (L - L) = 0$$

$$\tau_{xy} = 0$$

, for the vertical edge,

$$\sigma_x = 0$$

$$\sigma_y = \rho g (y - L)$$

$$\tau_{xy} = 0$$

, and for the bottom edge,

$$\sigma_x = 0$$

$$\sigma_y = \rho g(0 - L) = -\rho g L$$

$$\tau_{xy} = 0$$

Determine the displacement $u = u(x, y)$ and $v = v(x, y)$.

From the stress-strain relations,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} 0 \\ \rho g(y-L) \\ 0 \end{Bmatrix}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\rho g \frac{\nu}{E}(y - L)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\rho g}{E}(y - L)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Thus, by integration, the displacement functions are in the form of

$$u = -\rho g \frac{\nu}{E}(y - L)x + f_y$$

$$v = \frac{\rho g}{E}\left(\frac{y^2}{2} - Ly\right) + f_x$$

Substituting $u = u(x, y)$ and $v = v(x, y)$ into $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$

$$\left[-\rho g \frac{\nu}{E}x + \frac{df_y}{dy}\right] + \left[\frac{df_x}{dx}\right] = 0$$

Grouping the terms,

$$\left[-\rho g \frac{\nu}{E}x + \frac{df_x}{dx}\right] + \left[\frac{df_y}{dy}\right] = 0$$

Each expression in the parentheses must be a constant. If not, we could vary x alone (or y alone) and violate the equality. Thus,

$$-\rho g \frac{\nu}{E}x + \frac{df_x}{dx} = a_1$$

$$\frac{df_y}{dy} = -a_1$$

and $$f_x = \int\left(a_1 + \rho g \frac{\nu}{E}x\right)dx = a_1 x + \rho g \frac{\nu}{E}\frac{x^2}{2} + a_2$$

$$f_y = -\int a_1 dy = -a_1 y + a_3$$

Rewriting the displacement functions, we get

$$u = -\rho g \frac{\nu}{E}(y - L)x - a_1 y + a_3$$

$$v = \frac{\rho g}{E}(\frac{y^2}{2} - Ly) + a_1 x + \rho g \frac{\nu}{E}\frac{x^2}{2} + a_2 = \frac{\rho g}{E}\left[\frac{y^2}{2} - Ly + \frac{\nu}{2}x^2\right] + a_1 x + a_2$$

By using the boundary conditions of the block and the symmetry, at $x = 0$ and $y = 0$, $u = 0$ and $v = 0$, we obtain

$$a_2 = 0 \text{ and } a_3 = 0$$

At $x = 0$ and $y = 0$, $\frac{\partial v}{\partial x} = 0$,

$$\frac{\partial v}{\partial x} = \frac{\rho g \nu}{E} x + a_1 = 0$$

$$a_1 = -\frac{\rho g \nu}{E} x$$

Hence, we get the displacement functions in the form of

$$u = -\rho g \frac{\nu}{E}(y - L - 1)x$$

$$v = \frac{\rho g}{E}\left[\frac{y^2}{2} - Ly + \frac{\nu}{2}x^2 - \nu x\right]$$

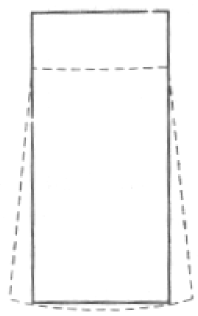


Fig. Ex 3-4b

The deformed shape of the block is shown in Fig. Ex 3-4b. It should be noted that

1.) The block shortens vertically and become wider toward the base due to Poisson's effect.
2.) All right angles are preserved since $\gamma_{xy} = 0$.
3.) The deflection at the base is incompatible with the rigid horizontal support
4.) According to the Saint-Venant's principle, the solution should be exact for $y > 2c$.

3.4 Solution by Polynomials

For the plane problems with long rectangular strip and the body forces are absent, the solutions of the biharmonic differential operator $\nabla^4 F = 0$ in the form of polynomial are of interest. By considering polynomials with various degrees and suitably adjusting their coefficients, a number of practical problems can be solved.

A quadratic polynomial is the lowest order polynomials that yield nonzero stresses from an Airy stress function. Consider the function

$$F = \frac{a_2}{2}x^2 + b_2 xy + \frac{c_2}{2}y^2$$

where a_2, b_2, and c_2 are constants. This Airy stress function is satisfied the equation $\nabla^4 F = 0$ and the stress components in this case are

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} = c_2$$

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} = a_2$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = -b_2$$

These stress boundary conditions are constant throughout the body. Thus, the stress function $F = \frac{a_2}{2}x^2 + b_2 xy + \frac{c_2}{2}y^2$ represents a combination of uniform tensions or compressions in two perpendicular directions and a uniform shear as shown in Fig. 3.3.

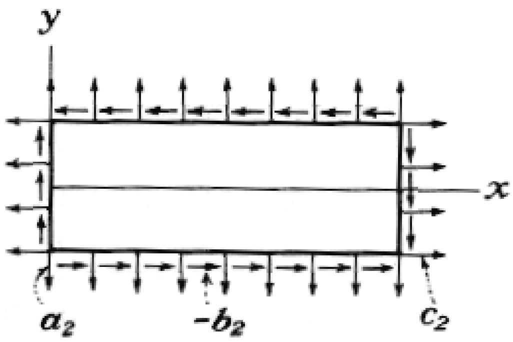


Fig 3.3

Let consider a stress function in the form of a cubic function.

$$F = \frac{a_3}{3(2)}x^3 + \frac{b_3}{2}x^2 y + \frac{c_3}{2}xy^2 + \frac{d_3}{3(2)}y^3$$

where a_3, b_3, c_3, and d_3 are constants. This Airy stress function is satisfied the equation $\nabla^4 F = 0$ and the stress components in this case are

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} = c_3 x + d_3 y$$

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} = a_3 x + b_3 y$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = -b_3 x - c_3 y$$

If all coefficients except d_3 are zero and $d_3 = 6a_1 =$ a constant,

$$\sigma_x = 6a_1 y \qquad \sigma_y = 0 \qquad \tau_{xy} = 0$$

The meaning and the usefulness of the obtained stress boundary condition depend on the region that we are choosing to consider. Let us consider Fig. 3-4.

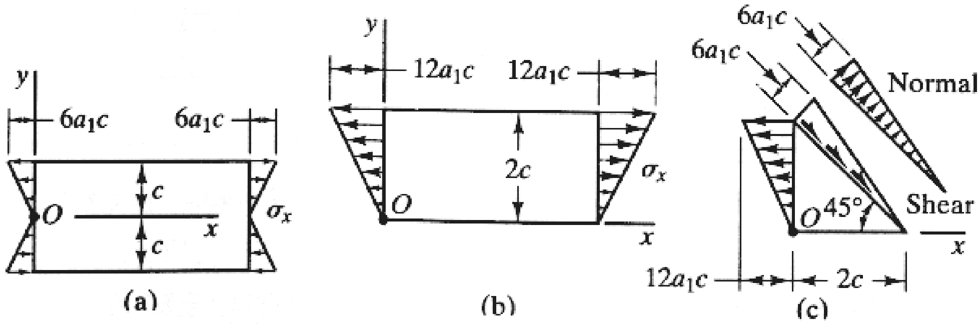


Fig. 3.4

1. If we choose the region as shown in Fig. 3.4a, the stress boundary condition represents a state of the normal stresses due to a pure bending applied at the ends of the beam.
2. If we choose the region as shown in Fig. 3.4b, the stress boundary condition represents a state of the normal stresses due to bending plus axial load applied to the ends of the beam.
3. If we choose the region as shown in Fig. 3.4c, the solution has no practical interest.

In taking the stress function in the form of quadratic and cubic polynomial equations, we are completely free in choosing the magnitudes of the coefficients, since the equation $\nabla^4 F = 0$ is always satisfied whatever values they may have. In the case of polynomials of higher degrees, the equation $\nabla^4 F = 0$ is satisfied only if certain relations between the coefficients are satisfied.

If all coefficients except b_3 are zero,

$$\sigma_x = 0 \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2} = b_3 y \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = -b_3 x$$

The stress boundary conditions in this case are shown in Fig. 3-5.

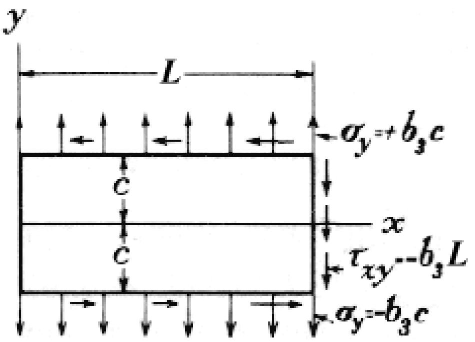


Fig 3.5

Let us consider the stress function in the form of a polynomial of the fourth degree,

$$F = \frac{a_4}{4(3)}x^4 + \frac{b_4}{3(2)}x^3 y + \frac{c_4}{2}x^2 y^2 + \frac{d_4}{3(2)}xy^3 + \frac{e_4}{4(3)}y^4$$

We can find that the function is satisfied the equation $\nabla^4 F = 0$ only if

$$e_4 = -(2c_4 + a_4)$$

The stress components in this case are

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} = c_4 x^2 + d_4 xy - (2c_4 + a_4)y^2$$

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} = a_4 x^2 + b_4 xy + c_4 y^2$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = -\frac{b_4}{2}x^2 - 2c_4 xy - \frac{d_4}{2}y^2$$

If all coefficients except d_4 are zero and $d_4 = 6\phi_4$ = a constant,

$$\sigma_x = 6\phi_4 xy \qquad \sigma_y = 0 \qquad \tau_{xy} = -3\phi_4 y^2$$

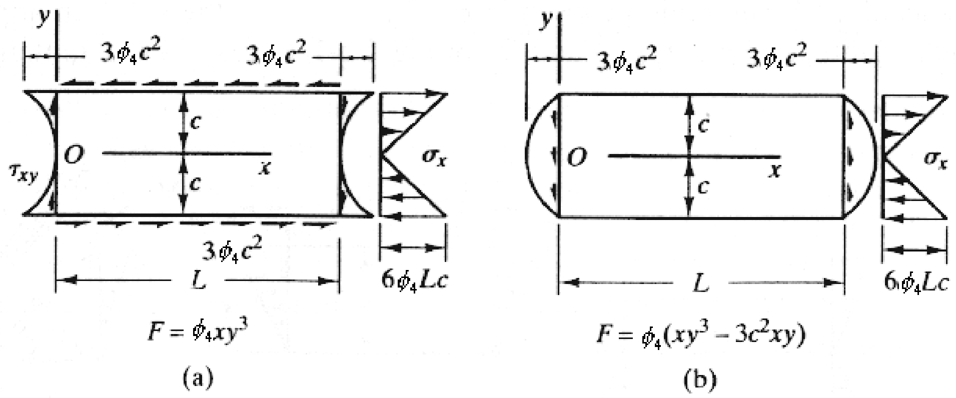


Fig. 3.6

Fig. 3.6a shows the stress boundary condition associated with the solution on the chosen region. The solution appears to lack practical interest.

Now, let us remove the shear stress on $y = \pm c$ by superimposing the stress function $F = -3\phi_4 c^2 xy$ on the previous stress function.

$$F = \phi_4 (xy^3 - 3c^2 xy)$$

This stress function is satisfied the equation $\nabla^4 F = 0$ and the stress components are

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} = 6\phi_4 xy$$

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} = 0$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = 3\phi_4 (c^2 - y^2)$$

Fig.3.6b shows the stress boundary condition associated with the solution on the chosen region. They appear to be the stress distribution corresponds to a cantilever beam having fixed support at $x = L$ and is subjected to both a parabolic distribution of τ_{xy} and pure bending at $x = 0$. Note that, according to the Saint-Venant's principle, the obtained elasticity solution must be considered as approximate near the fixed support and loading point ($2c < x < L - 2c$) due to the Poisson's effect and stress concentration.

$$P = \int_{-c}^{c} \tau_{xy} dy = 3\phi_4 \left[2c^3 - \frac{2c^3}{3} \right] = 4\phi_4 c^3$$

$$\sigma_x = \frac{My}{I} = \frac{(Px)y}{1(2c)^3/12} = 6\phi_4 xy$$

3.5 End Effects

In some problems in which the structure is subjected to loads producing stress concentrations in the area of loading point, we need to use the Saint-Venant's principle to simplify the problems by replacing the loads with a statically equivalent load.

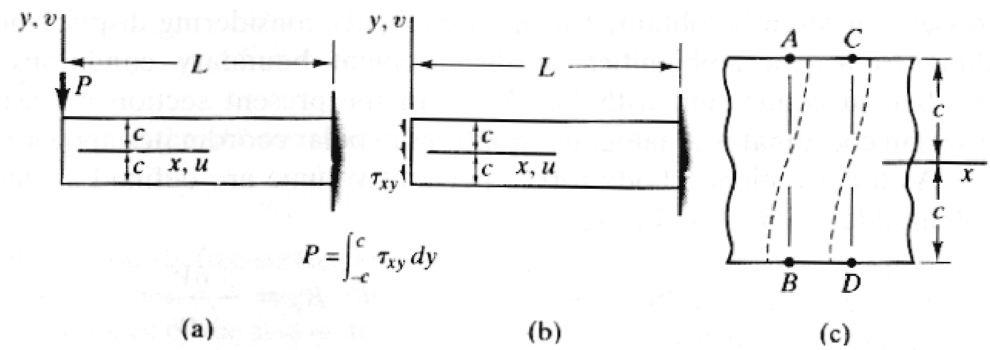


Fig 3.7

Consider a cantilever beam having a unit width as shown in Fig. 3.7a. The concentrated load P makes the problem difficult to solve since it is difficult to satisfy the force boundary conditions at the loading point. Hence, the load P is replaced by a quadratic distribution of surface traction $\Phi_y = -\tau_{xy}$ that is zero at $y = \pm c$ and maximum at $y = 0$ as shown in Fig. 3.7b. According to the Saint-Venant's principle, these two different loading patterns practically produce no different effect to the beam at a distance of $2c$ or more.

In addition, the displacement boundary condition at the fixed end of the beam is $u = 0$ and $v = 0$ for $-c < y < c$. This condition makes the elasticity solution difficult to obtain. To make the problem easier, we assume the initial vertical line at x to warp into the shapes as shown in Fig. 3.7c. This deformation is due to the shear stress τ_{xy}. Due to the fixed support at $x = L$, we may

1. Set $\dfrac{\partial v}{\partial x} = 0$ at $y = 0$ to make the beam axis horizontal at $x = L$.
2. Set $\dfrac{\partial u}{\partial y} = 0$ at $y = 0$ to make the beam vertical line vertical at $x = L$.

3.6 Determination of Displacements from Stresses

If the stress functions are known, we can determine the displacement by integration of the strain-displacement relations as shown in the following example.

Example 3-5: Bending of a cantilevered beam loaded at the end

Determine the displacements of a cantilevered beam having a unit width and loaded at the end as shown in Fig. 3.7a.

Consider the cantilevered beam having a unit width as shown in Fig. 3.8a. The upper and lower edges are free from load. As discussed before, the load P is replaced by a quadratic distribution of surface traction $\Phi_y = -\tau_{xy}$ that is zero at $y = \pm c$ and maximum at $y = 0$. This condition can be satisfied by using the stress components found by using the Airy stress function in the form of

$$\sigma_x = 6\phi_4 xy \qquad \sigma_y = 0 \qquad \tau_{xy} = 3\phi_4(c^2 - y^2)$$

The constant ϕ_4 can be found by using the force boundary condition: sum of the shearing force distribution over the end of the beam must be equal to the load P.

$$P = \int_{-c}^{c} \tau_{xy} dy = 3\phi_4\left[2c^3 - \frac{2c^3}{3}\right] = 4\phi_4 c^3$$

$$\phi_4 = \frac{P}{4c^3}$$

Since the moment of inertial of the cross section having a unit thickness is $\frac{2c^3}{3}$, the stress components equations can be written as

$$\sigma_x = \frac{3P}{2c^3}xy = \frac{Pxy}{I}$$

$$\tau_{xy} = \frac{3P}{4c}(1 - \frac{y^2}{c^2}) = \frac{P}{2I}(c^2 - y^2)$$

These stresses can also be obtained directly by using the method of mechanics of material where

$$\sigma_x = \frac{My}{I} \text{ and } \tau_{xy} = \frac{VQ}{It}$$

Now, let determine the displacement corresponding to the stresses. Using Hooke's law for plane stress, we have

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} Pxy/I \\ 0 \\ P(c^2 - y^2)/(2I) \end{Bmatrix} = \begin{Bmatrix} Pxy/(EI) \\ -\nu Pxy/(EI) \\ P(c^2 - y^2)/(2IG) \end{Bmatrix}$$

The strain-displacement relations of the plane stress are

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{Pxy}{EI} \qquad \varepsilon_y = \frac{\partial v}{\partial y} = -\frac{\nu Pxy}{EI} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{P}{2IG}(c^2 - y^2)$$

Integrating ε_x and ε_y, we have

$$u = \frac{Px^2 y}{2EI} + f_y \qquad v = -\frac{\nu Pxy^2}{2EI} + f_x$$

where f_y and f_x are unknown functions of y only and x only, respectively. They may not be a constant since we are dealing with partial derivative. Substituting the values of u and v into the shear strain-displacement relation,

$$\gamma_{xy} = \frac{Px^2}{2EI} + \frac{df_y}{dy} - \frac{\nu Py^2}{2EI} + \frac{df_x}{dx} = \frac{P}{2IG}(c^2 - y^2)$$

$$\left[\frac{Px^2}{2EI} + \frac{df_x}{dx}\right] + \left[-\frac{\nu Py^2}{2EI} + \frac{df_y}{dy} + \frac{Py^2}{2IG}\right] = \frac{Pc^2}{2IG}$$

In this equation, we can see that $F(x) = \left[\frac{Px^2}{2EI} + \frac{df_x}{dx}\right]$ are functions of x only, $G(y) = \left[-\frac{\nu Py^2}{2EI} + \frac{df_y}{dy} + \frac{Py^2}{2IG}\right]$ are functions of y only, and $K = \frac{Pc^2}{2IG}$ is independent of x and y.

$$F(x) + G(y) = K$$

This equation means that $F(x)$ must be some constant A and $G(y)$ must be some constant B. If not, $F(x)$ and $G(y)$ would vary with x and y, respectively. In addition, by varying x alone or y alone, the equality would be violated. Hence,

$$A + B = \frac{Pc^2}{2IG}$$

$$\frac{df_x}{dx} = A - \frac{Px^2}{2EI}$$

$$\frac{df_y}{dy} = B + \frac{\nu Py^2}{2EI} - \frac{Py^2}{2IG}$$

Integrating $\frac{df_x}{dx}$ and $\frac{df_y}{dy}$, we have

$$f_x = -\frac{Px^3}{6EI} + Ax + C$$

$$f_y = \frac{\nu Py^3}{6EI} - \frac{Py^3}{6IG} + By + D$$

Substituting into the expressions of u and v, we have

$$u = \frac{Px^2 y}{2EI} + \frac{\nu Py^3}{6EI} - \frac{Py^3}{6IG} + By + D$$

$$v = -\frac{\nu Pxy^2}{2EI} - \frac{Px^3}{6EI} + Ax + C$$

Assuming that the centroid of the cross section is fixed. Then, the expressions of u and v are zero at $x = L$ and $y = 0$.

$$D = 0$$

$$C = \frac{PL^3}{6EI} - AL$$

The deflection curve of the beam at $y = 0$ is

$$v_{y=0} = -\frac{Px^3}{6EI} + \frac{PL^3}{6EI} - A(L - x)$$

From the discussion about the end effects, due to the fixed support at $x = L$, we may

1. Set $\dfrac{\partial v}{\partial x} = 0$ at $y = 0$ to make the beam axis horizontal at $x = L$.
2. Set $\dfrac{\partial u}{\partial y} = 0$ at $y = 0$ to make the beam vertical line vertical at $x = L$.

For the first case $\dfrac{\partial v}{\partial x} = 0$ at $y = 0$ and $x = L$,

$$\frac{\partial v}{\partial x} = -\frac{PL^2}{2EI} + A = 0$$

$$A = \frac{PL^2}{2EI}$$

Then, from the equation $A + B = \dfrac{Pc^2}{2IG}$,

$$B = \frac{Pc^2}{2IG} - \frac{PL^2}{2EI}$$

Substituting all constants $A = \dfrac{PL^2}{2EI}$, $B = \dfrac{Pc^2}{2IG} - \dfrac{PL^2}{2EI}$, $C = \dfrac{PL^3}{6EI} - AL = -\dfrac{PL^3}{3EI}$, $D = 0$ into the expressions of u and v, we have

$$u = \frac{Px^2y}{2EI} + \frac{\nu Py^3}{6EI} - \frac{Py^3}{6IG} + \left[\frac{Pc^2}{2IG} - \frac{PL^2}{2EI}\right]y$$

$$v = -\frac{\nu Pxy^2}{2EI} - \frac{Px^3}{6EI} + \frac{PL^2x}{2EI} - \frac{PL^3}{3EI}$$

The deflection curve of the beam at $y = 0$ can be rewritten as

$$v_{y=0} = -\frac{Px^3}{6EI} + \frac{PL^2x}{2EI} - \frac{PL^3}{3EI}$$

At $x = 0$, the deflection of the beam at $y = 0$ is

$$v_{\substack{y=0\\x=0}} = -\frac{PL^3}{3EI}$$

which is identical to the one obtained by using the mechanics of materials method.

To show the warping of the beam cross section produced by the shearing stress, let us consider the horizontal displacement u at the support $x = L$.

$$u_{x=L} = \frac{\nu Py^3}{6EI} - \frac{Py^3}{6IG} + \frac{Pc^2 y}{2IG}$$

$$\left.\frac{\partial u}{\partial y}\right|_{x=L} = \frac{\nu Py^2}{2EI} - \frac{Py^2}{2IG} + \frac{Pc^2}{2IG}$$

$$\left.\frac{\partial u}{\partial y}\right|_{\substack{x=L\\y=0}} = \frac{Pc^2}{2IG} = \frac{3P}{4cG}$$

This rotation of the cross section is due to the shearing stress $\tau_{xy} = \dfrac{3P}{4c}$ and having the clockwise direction.

Consider the second case where $\dfrac{\partial u}{\partial y} = 0$ at $y = 0$ to make the beam vertical line vertical at $x = L$, we have

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\frac{Px^2 y}{2EI} + \frac{\nu Py^3}{6EI} - \frac{Py^3}{6IG} + By + D\right] = 0$$

$$\frac{Px^2}{2EI} + \frac{\nu Py^2}{2EI} - \frac{Py^2}{2IG} + B = 0$$

$$B = -\frac{Px^2}{2EI} - \frac{\nu Py^2}{2EI} + \frac{Py^2}{2IG}$$

At $y = 0$ and $x = L$,

$$B = -\frac{PL^2}{2EI}$$

Then, from the equation $A + B = \dfrac{Pc^2}{2IG}$,

$$A = \frac{Pc^2}{2IG} + \frac{PL^2}{2EI}$$

Substituting into the expressions of v at $y = 0$ with $C = \dfrac{PL^3}{6EI} - AL = -\dfrac{Pc^2 L}{2IG} - \dfrac{PL^3}{3EI}$, we have

$$v_{y=0} = -\frac{Px^3}{6EI} + \frac{Pc^2 x}{2IG} + \frac{PL^2 x}{2EI} - \frac{Pc^2 L}{2IG} - \frac{PL^3}{3EI}$$

$$v_{y=0} = -\frac{Px^3}{6EI} - \frac{Pc^2 x}{2IG}(L-x) + \frac{PL^2 x}{2EI} - \frac{PL^3}{3EI}$$

Comparing this equation with the equation $v_{y=0} = -\frac{Px^3}{6EI} + \frac{PL^2 x}{2EI} - \frac{PL^3}{3EI}$ previously obtained, we have the deflection of the cantilevered beam increased by

$$\frac{Pc^2 x}{2IG}(L-x) = \frac{3P}{4cG}(L-x)$$

This term is an estimate of the effect of shearing force on the deflection of the beam. In reality, the cross-section near the fixed support is not free to rotate. Thus, the distribution of stresses is different from the obtained results. However, if the beam is long compared to the depth, the results are satisfactory.

3.7 Plane Stress Problems in Polar Coordinate

The polar coordinate as shown in Fig. 3.8 is useful in the stress analysis of the structures such as curved beams, circular rings and disks, pressurized cylinder.

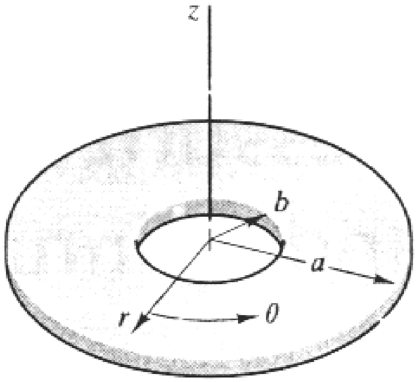


Fig. 3.8

Equilibrium Equations in Polar Coordinate

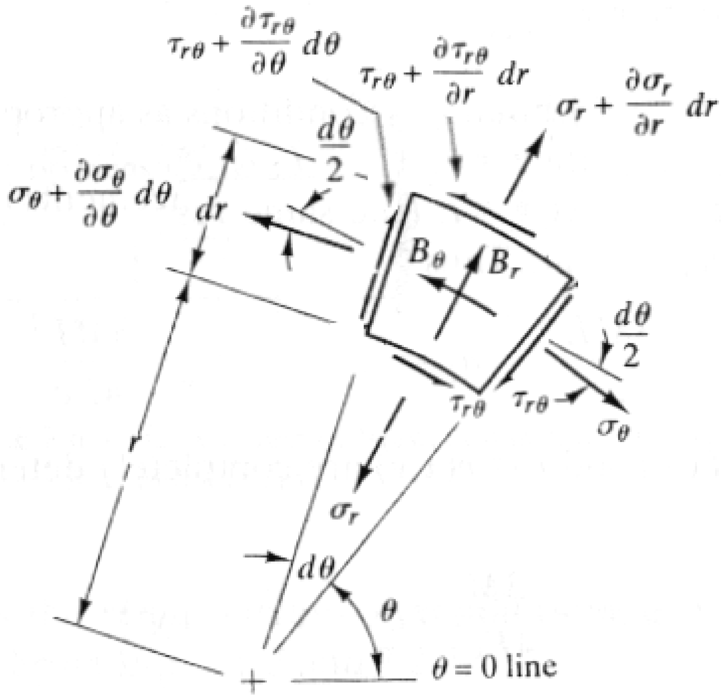


Fig. 3.9

Consider an infinitesimal small element subjected to the state of stresses in the polar coordinate as shown in Fig. 3.9. If this element is in equilibrium, we have the summations of the forces in the radial direction and circumferential direction are equal to zero.

$\sum F_r = 0;$

$$-\sigma_r(rd\theta) - \tau_{r\theta}(dr) - \sigma_\theta \sin\frac{d\theta}{2}(dr) + \left[\sigma_r + \frac{\partial \sigma_r}{\partial r}dr\right](r+dr)d\theta +$$

$$\left[\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta}d\theta\right]dr - \left[\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta}d\theta\right]\sin\frac{d\theta}{2}(dr) + B_r r dr d\theta = 0$$

$\sum F_{\theta} = 0;$

$$-\tau_{r\theta}(rd\theta) - \sigma_{\theta}(dr) - \tau_{r\theta}\sin\frac{d\theta}{2}(dr) + \left[\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r}dr\right](r+dr)d\theta +$$

$$\left[\sigma_{\theta} + \frac{\partial \sigma_{\theta}}{\partial \theta}d\theta\right]dr - \left[\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta}d\theta\right]\sin\frac{d\theta}{2}(dr) + B_{\theta}rdrd\theta = 0$$

Expanding the above two equations, setting $\sin\frac{d\theta}{2} = \frac{d\theta}{2}$, and neglecting the higher-order terms, we have the equilibrium equations in the polar coordinate.

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + B_r = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + 2\frac{\tau_{r\theta}}{r} + B_\theta = 0$$

Airy Stress Function

To solve the plane problems in polar coordinate, we begin by finding stresses that satisfy the equations of equilibrium. However, the two equilibrium equations have three unknowns stresses. Hence, The number of the possible stress solutions is infinite.

By using coordinate transformation from the Cartesian coordinates (x, y) to the polar coordinates (r, θ), we have

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

By taking the inverse of the matrix, we have

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{Bmatrix}$$

The relations between the Cartesian coordinates (x, y) to the polar coordinates (r, θ) are

$$r^2 = x^2 + y^2$$

$$\theta = \arctan\frac{y}{x}$$

The derivatives of the relation with respect to x and y are

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial\theta}{\partial x}=-\frac{y}{r^2}=-\frac{\sin\theta}{r} \qquad \frac{\partial\theta}{\partial y}=\frac{x}{r^2}=\frac{\cos\theta}{r}$$

Thus, for any function $f(x,y)$ that can be written in the polar coordinate as $f(r\cos\theta, r\sin\theta)$, we have the first partial derivative of the function $f(x,y)$ as

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial x}=\cos\theta\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}$$

Also, the second partial derivative of the function $f(x,y)$ can be written as

$$\frac{\partial^2 f}{\partial x^2}=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right)$$

$$\frac{\partial^2 f}{\partial x^2}=\cos^2\theta\frac{\partial^2 f}{\partial r^2}-\sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial f}{\partial r}\right)$$

$$+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)$$

Rearranging the expressions,

$$\frac{\partial^2 f}{\partial x^2}=\cos^2\theta\frac{\partial^2 f}{\partial r^2}-\sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)-\sin\theta\cos\theta\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right)+\sin\theta\cos\theta\frac{1}{r^2}\frac{\partial f}{\partial\theta}+$$

$$\sin^2\theta\left[\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\right]$$

Since the term $\frac{1}{r^2}\frac{\partial f}{\partial\theta}=\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial\theta}\right)-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)$ and $\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial\theta}\right)$, thus, we have

$$\frac{\partial^2 f}{\partial x^2}=\cos^2\theta\frac{\partial^2 f}{\partial r^2}-2\sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)+\sin^2\theta\left(\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\right)$$

Similarly, we find

$$\frac{\partial^2 f}{\partial y^2}=\sin^2\theta\frac{\partial^2 f}{\partial r^2}+2\sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)+\cos^2\theta\left(\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\right)$$

$$-\frac{\partial^2 f}{\partial x\partial y}=\sin\theta\cos\theta\left(\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}-\frac{\partial^2 f}{\partial r^2}\right)-(\cos^2\theta-\sin^2\theta)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)$$

To transform the Airy stress function in the Cartesian coordinates $F=F(x,y)$ to the Airy stress function in the polar coordinates $F=F(r,\theta)$, we replace the function $f(x,y)$ by the Airy stress function in the Cartesian coordinates $F=F(x,y)$. Neglecting the body forces, we have

$$\sigma_x=\frac{\partial^2 F}{\partial y^2} \qquad \sigma_y=\frac{\partial^2 F}{\partial x^2} \qquad \tau_{xy}=-\frac{\partial^2 F}{\partial x\partial y}$$

$$\sigma_x = \sin^2\theta\frac{\partial^2 F}{\partial r^2} + 2\sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right) + \cos^2\theta\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right)$$

$$\sigma_y = \cos^2\theta\frac{\partial^2 F}{\partial r^2} - 2\sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right) + \sin^2\theta\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right)$$

$$\tau_{xy} = \sin\theta\cos\theta\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2} - \frac{\partial^2 F}{\partial r^2}\right) - (\cos^2\theta - \sin^2\theta)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)$$

Substituting σ_x, σ_y, and τ_{xy} into the coordinate transformation matrix,

$$\begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \tau_{r\theta}\end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn\\ n^2 & m^2 & -2mn\\ -mn & mn & m^2-n^2\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}$$

, and rearranging the terms, we have

$$\sigma_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}$$

$$\sigma_\theta = \frac{\partial^2 F}{\partial r^2}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)$$

Substituting the obtained functions of the stresses into the equilibrium equations, $\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0$ and $\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + 2\frac{\tau_{r\theta}}{r} = 0$, and neglecting the body forces, we can see that the stress functions are satisfied the equations of equilibrium. Thus, the assumed stress functions constitute a general solution.

Now, the problem is reduced to finding a function $F = F(r,\theta)$ that satisfies the compatibility conditions and the boundary conditions.

Next, we will transform the compatibility equation in the Cartesian coordinates, $\nabla^4 F = \left[\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2\partial y^2} + \frac{\partial^4 F}{\partial y^4}\right] = 0$, to the polar coordinates. Consider the harmonic operator of any function $f(x,y)$ that can be written in the polar coordinate as $f(r\cos\theta, r\sin\theta)$.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f = \cos^2\theta\frac{\partial^2 f}{\partial r^2} - 2\sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) + \sin^2\theta\left(\frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\right)$$

$$+\sin^2\theta\frac{\partial^2 f}{\partial r^2} + 2\sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) + \cos^2\theta\left(\frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\right)$$

Thus,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)f$$

This expression tells us that the harmonic operator on the left-handed side, which is in the Cartesian coordinates, is equivalent to the operator on the right-handed side, which is in the polar coordinates.

The strain compatibility relation of the plane stress which is in the Cartesian coordinates is

$$\frac{\partial^2 \varepsilon_y}{\partial x^2}+\frac{\partial^2 \varepsilon_x}{\partial y^2}=\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Substituting the strain-stress relations which are

$$\varepsilon_x=\frac{1}{E}(\sigma_x-\nu\sigma_y) \qquad \varepsilon_y=\frac{1}{E}(\sigma_y-\nu\sigma_x) \qquad \gamma_{xy}=\frac{2(1+\nu)}{E}\tau_{xy}$$

into the strain compatibility relation, we have

$$\frac{\partial^2}{\partial x^2}(\sigma_y-\nu\sigma_x)+\frac{\partial^2}{\partial y^2}(\sigma_x-\nu\sigma_y)=2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

Differentiating the equilibrium equations $\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0$ with respect to x and $\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0$ with respect to y, and adding them, we find

$$-\frac{\partial^2 \sigma_x}{\partial x^2}-\frac{\partial^2 \sigma_y}{\partial y^2}=2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

Then, substituting it into the strain compatibility relation, we have

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=0$$

By adding the stress transformation σ_x and σ_y from the transformation matrix $\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}m^2 & n^2 & -2mn\\ n^2 & m^2 & 2mn\\ mn & -mn & m^2-n^2\end{bmatrix}\begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \tau_{r\theta}\end{Bmatrix}$, we have

$$\sigma_x+\sigma_y=\sigma_r+\sigma_\theta$$

Consequently, the polar coordinate form of the compatibility equation in term of stress when the body forces are absent is

$$\nabla^2(\sigma_r+\sigma_\theta)=0$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_r+\sigma_\theta)=0$$

Since $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)f$, we have

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_r+\sigma_\theta)=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\sigma_r+\sigma_\theta)$$

Thus, the transformation of the harmonic differential operator ∇^2 to the polar coordinate gives

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

or

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Since $\nabla^4 F=\nabla^2(\nabla^2 F)$, therefore, we have

$$\nabla^4 F=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 F}{\partial r^2}+\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)=0$$

Strain Components in Polar Coordinates

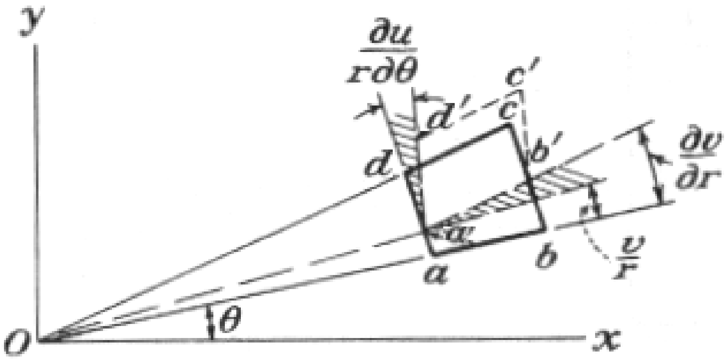


Fig. 3.10

Let $u=u(r,\theta)$ and $v=v(r,\theta)$ are the radial and tangential (circumferential) components of the displacement in the polar coordinates as shown in the Fig. 3.10. If the radial displacement of the side ad is u, and of the side bc is $u+\frac{\partial u}{\partial r}dr$,

$$\varepsilon_r=\frac{\partial u}{\partial r}$$

The strain in the tangential direction depends not only on the tangential displacement v but on the radial direction as well. Assuming that the points a and d have only the displacement u, the new length of the arc ad is $(r+u)d\theta$, then, the tangential strain due to only the displacement u is

$$\frac{(r+u)d\theta - rd\theta}{rd\theta} = \frac{u}{r}$$

The difference in the tangential displacement of the sides ab and cd is $\frac{\partial v}{\partial \theta}d\theta$. The tangential strain due to only the displacement v is

$$\frac{\partial v}{\partial \theta}d\theta\frac{1}{rd\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

The total tangential strain is

$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

To determine the shearing strain $\gamma_{r\theta}$, let the element $abcd$ deforms to the position $a'b'c'd'$. The angle between ad and $a'd'$ is

$$\frac{\partial u}{\partial \theta}d\theta\frac{1}{rd\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta}$$

The angle between ab and $a'b'$ is $\frac{\partial v}{\partial r}$. This angle must be subtracted by the rigid body rotation $\frac{v}{r}$ about the axis passing through point O. This rigid body rotation does not contribute to the shearing strain $\gamma_{r\theta}$. Thus, the total shearing strain $\gamma_{r\theta}$ is

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

Stress-strain Relations

Since the stress-strain relations are derived based on the internal energy in the rectangular coordinate system, they also valid for other *orthogonal coordinate* systems such as cylindrical or spherical coordinates. Thus, the stress-strain relations in the polar coordinate can be written by changing the subscript x to r (radial) and y to θ (circumferential).

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{Bmatrix}$$

3.8 Stress Distribution Symmetrical about an Axis: Pressurized Cylinder

Consider a circular plate subjected to the internal and external pressure as shown in Fig. 3.11. The plate may be a slice of a long thick-walled cylinder structure. It has an internal radius of a and external radius of b. Due to the symmetry of the plate and the loading condition, the stress components occurred in the small plate element within the radius a and

b depend on the distance from the center of the plate r only. Let the center of the plate is the origin of the polar coordinates (r, θ). Thus, the compatibility equation becomes

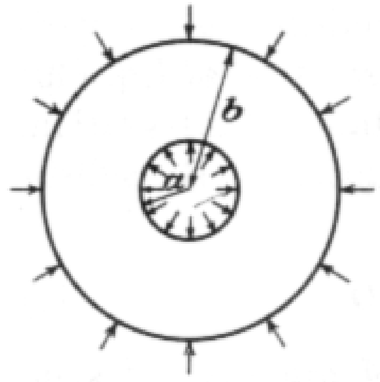


Fig. 3.11

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 F}{\partial r^2}+\frac{1}{r}\frac{\partial F}{\partial r}\right)=0$$

By expanding the equation, we find

$$\frac{d^4F}{dr^4}+\frac{2}{r}\frac{d^3F}{dr^3}+\frac{1}{r^2}\frac{d^2F}{dr^2}+\frac{1}{r^3}\frac{dF}{dr}=0$$

This is ordinary differential equation. It can be reduced to a linear differential equation with constant coefficients by introducing a new variable t such as $r=e^t$. Then, the general solution of the equation is in the form of

$$F=A\log r+Br^2\log r+Cr^2+D$$

This solution has four unknowns constants of integration which can be found by using the boundary conditions. The correctness of the equation can be verified by substituting back into the compatibility equation.

The corresponding stress components are

$$\sigma_r=\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}=\frac{1}{r}\frac{\partial F}{\partial r}=\frac{A}{r^2}+B(1+2\log r)+2C$$

$$\sigma_\theta=\frac{\partial^2 F}{\partial r^2}=-\frac{A}{r^2}+B(3+2\log r)+2C$$

$$\tau_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right)=0$$

If the plate has no hole at the origin of the coordinates ($r=0$), the constant A and B must be zero. This is due to the fact that when $r\to 0$, the stresses $\sigma_r\to\alpha$ and $\sigma_\theta\to\alpha$ which are physically impossible. Then, we have $\sigma_r=\sigma_\theta=2C=\text{constant}$ and the plate is in a condition of uniform tension or uniform compression.

If the plate has the hole at the origin, we need to set the constants B to be zero (see Theory of Elasticity, Timoshenko, p.78) in order to find the condition of stresses in the plate.

$$\sigma_r = \frac{A}{r^2} + 2C$$

$$\sigma_\theta = -\frac{A}{r^2} + 2C$$

These expressions represent the stress distribution in a hollow cylinder subjected to uniform pressure on the inner and outer surfaces. From the Fig. 3-11, let the uniform inner and uniform outer pressures are p_i and p_o. Then, the boundary conditions are

$$(\sigma_r)_{r=a} = \frac{A}{a^2} + 2C = -p_i$$

$$(\sigma_r)_{r=b} = \frac{A}{b^2} + 2C = -p_o$$

Solving the simultaneous for A and $2C$, we have

$$A = \frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2}$$

$$2C = \frac{p_i a^2 - p_o b^2}{b^2 - a^2}$$

Therefore, the stress components are

$$\sigma_r = \frac{1}{r^2}\frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2}$$

$$\sigma_\theta = -\frac{1}{r^2}\frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2}$$

Rearranging the equations, we obtain

$$\sigma_r = p_i \frac{a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) - p_o \frac{b^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = p_i \frac{a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) - p_o \frac{b^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right)$$

These stress components are occurred in thick-walled cylinder subjected to the uniform inner pressure p_i and uniform outer pressures p_o. The distribution of the stress components across the thickness of the cylinder is shown in Fig. 3.12a and 3-12b for the case of internal pressure only and external pressure only, respectively.

Thick-walled Cylinder without External Pressure

If the external pressure $p_o = 0$, the cylinder is subjected to internal pressure only.

$$\sigma_r = p_i \frac{a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_\theta = p_i \frac{a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

It can be seen that the circumferential stress is always larger than the radial stress at the same value of r.

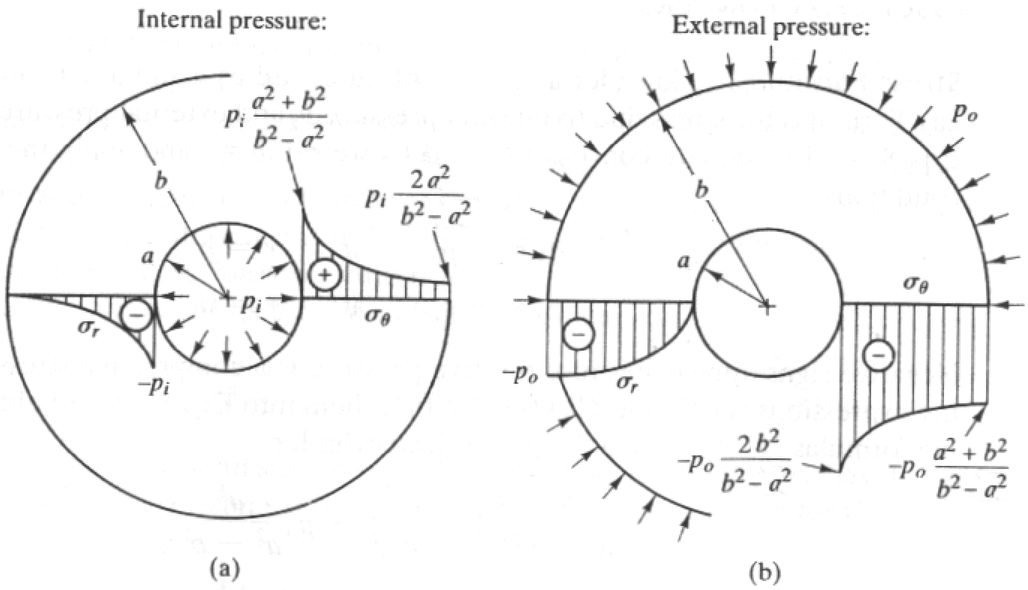


Fig. 3.12

If the cylinder as shown in Fig. 3.13 has end cap, the axial stress σ_z presents in the cylinder. The force acting on the end cap due to the internal pressure p_i is $p_i\pi a^2$. The reaction force on the wall of the cylinder is $\sigma_z\pi(b^2 - a^2)$. Thus,

$$\sigma_z = p_i \frac{a^2}{b^2 - a^2}$$

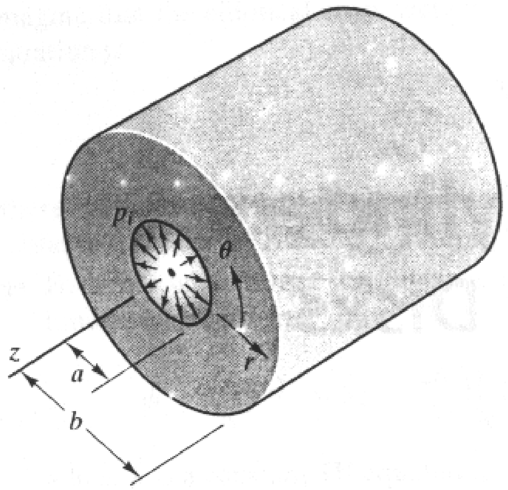


Fig. 3.13

The maximum shear stress in this case is occurred at $r = a$. The radial stress is the minimum principal stress, $\sigma_r = -p_i = \sigma_3$ and the circumferential stress is the maximum principal stress, $\sigma_\theta = p_i \dfrac{a^2 + b^2}{b^2 - a^2} = \sigma_1$. Therefore, the maximum shear stress is

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{p_i b^2}{b^2 - a^2}$$

Then, we can determine the radial and circumferential strains by using the strain-stress relation.

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu(\sigma_\theta + \sigma_z)\right]$$

$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right]$$

Finally, we can determine the displacements by using the strain-displacement relations.

$$\varepsilon_r = \frac{\partial u}{\partial r}$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

Thin-walled Cylinder without External Pressure

For thin-walled cylinder having the ratio of $a/(b-a) = a/t >> 20$ and $a/b \approx 1$, we have

$$\sigma_r = 0$$

$$\sigma_\theta = \frac{p_i a}{t}$$

$$\sigma_z = \frac{p_i a}{2t}$$

Example 3-6

A steel cylinder with end caps is to have the inner radius $a = 10$ mm and outer radius of $b = 31.3$ mm. Under the working pressure of 140 MPa, what is the radius expansion at $r = a$ and $r = b$? Use $E = 204$ GPa and $\nu = 0.29$.

At working pressure $p_i = 140$ MPa, we have the state of stresses at $r = a = 10$ mm as

$$\sigma_r = p_i \frac{a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) - p_o \frac{b^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_r = -140 \text{ MPa}$$

$$\sigma_\theta = p_i \frac{a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) - p_o \frac{b^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = 172 \text{ MPa}$$

$$\sigma_z = p_i \frac{a^2}{b^2 - a^2}$$

$$\sigma_z = 16 \text{ MPa}$$

Circumferential strain at $r = a = 10$ mm is

$$\varepsilon_{\theta a} = \frac{1}{204000}\left[172 - 0.29(16 - 140)\right] = 0.00102$$

Thus,

$$0.00102 = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Since the rate of change of v with respect to θ is zero due to the symmetry, the radius expansion at $r = a = 10$ mm is

$$u_a = 0.00102(10) = 0.0102 \text{ mm}$$

In the similar fashion, the radius expansion at $r = b = 31.3$ mm with $\sigma_{\theta b} = 31.8$ MPa is

$$u_b = b\varepsilon_{\theta b} = (31.3)\frac{1}{204000}\left[31.8 - 0.29(16 + 0)\right] = 0.0042 \text{ mm}$$

3.9 Effect of Circular Holes on Stress Distributions in Plates

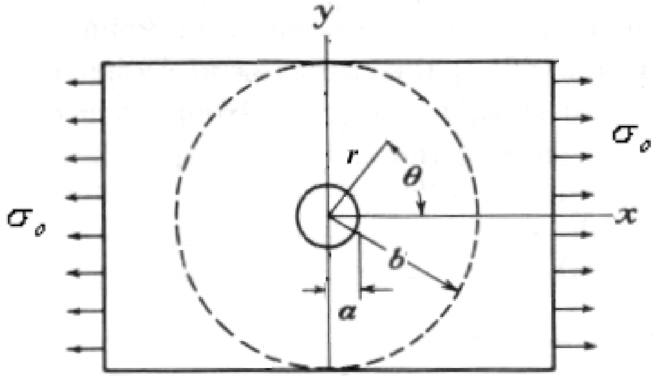


Fig. 3.14

Consider the plate having unit thickness and a small circular hole of radius a at the center of the plate as shown in Fig. 3.14. It is subjected to a uniformly distributed axial tensile stress σ_o in the $x-$direction.

The stress distribution in the area neighboring the hole will be different from the stress distribution in the plate without the hole. However, from the Saint-Venant's principle, the change is negligible at distances which are large compared with the radius of the hole, a.

Let the radius b is large in comparison to the radius a of the hole so that the stresses at the b are the same as in the plate without the hole. By using the stress transformation equation from the Cartesian coordinates (x, y) to the polar coordinates (r, θ),

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

and since $\sigma_x = \sigma_o$, $\sigma_y = 0$, and $\tau_{xy} = 0$, thus,

$$(\sigma_r)_{r=b} = \sigma_o \cos^2\theta = \frac{\sigma_o}{2}(1 + \cos 2\theta)$$

$$(\tau_{r\theta})_{r=b} = -\frac{\sigma_o}{2}\sin 2\theta$$

It can be seen that the stresses can be considered separately into two parts. The first part is due to the constant components of $\frac{\sigma_o}{2}$. The solution of this part is obtained in previous section as

$$\sigma_r = p_i \frac{a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) - p_o \frac{b^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = p_i \frac{a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) - p_o \frac{b^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right)$$

where $p_i = 0$ and $p_0 = -\dfrac{\sigma_o}{2}$.

The remaining part consists of the normal stress $\dfrac{\sigma_o}{2}\cos 2\theta$ and the shearing stress $-\dfrac{\sigma_o}{2}\sin 2\theta$. The solution of this part can be determined by using the Airy stress function of the form

$$F = f(r)\cos 2\theta$$

Substituting the function into the compatibility condition,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) = 0$$

Hence, we can find the following differential equation to determine $f(r)$.

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{4f}{r^2}\right) = 0$$

This is an ordinary differential equation. The general solution of the equation is in the form of

$$f(r) = Ar^2 + Br^4 + C\frac{1}{r^2} + D$$

Thus, the stress function is

$$F = \left(Ar^2 + Br^4 + C\frac{1}{r^2} + D\right)\cos 2\theta$$

The corresponding stress components are

$$\sigma_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{\partial^2 F}{\partial r^2} = \left(2A + 12Br^4 + \frac{6C}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right) = \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin 2\theta$$

By using the boundary conditions, we can determine the constants of integration. At the outer boundary where $r = b$, the remaining stress that produce the stress components $(\sigma_r)_{r=b}$ and $(\tau_{r\theta})_{r=b}$ are the normal stress $\dfrac{\sigma_o}{2}\cos 2\theta$ and the shearing stress $-\dfrac{\sigma_o}{2}\sin 2\theta$. Thus, the remaining stresses are $(\sigma_r)_{r=b} = \dfrac{\sigma_o}{2}\cos 2\theta$ and $(\tau_{r\theta})_{r=b} = -\dfrac{\sigma_o}{2}\sin 2\theta$.

$$\frac{\sigma_o}{2}\cos 2\theta = -\left(2A + \frac{6C}{b^4} + \frac{4D}{b^2}\right)\cos 2\theta$$

$$2A + \frac{6C}{b^4} + \frac{4D}{b^2} = -\frac{\sigma_o}{2}$$

$$-\frac{\sigma_o}{2}\sin 2\theta = \left(2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2}\right)\sin 2\theta$$

$$2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2} = -\frac{\sigma_o}{2}$$

At the edge of the hole where $r = a$, the stress components must be zero since there is free from external force.

$$0 = -\left(2A + \frac{6C}{a^4} + \frac{4D}{a^2}\right)\cos 2\theta$$

$$2A + \frac{6C}{a^4} + \frac{4D}{a^2} = 0$$

$$0 = \left(2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2}\right)\sin 2\theta$$

$$2A + 6Ba^2 - \frac{6C}{a^4} - \frac{2D}{a^2} = 0$$

Solving these four simultaneous equations and assuming that the plate is infinitely large, $a/b = 0$, we get

$$A = -\frac{\sigma_o}{4}$$

$$B = 0$$

$$C = -\frac{a^4}{4}\sigma_o$$

$$D = \frac{a^2}{2}\sigma_o$$

Substituting the constants of integration into the stress component equation and plus the stresses component due to the uniform tension stress $\frac{\sigma_o}{2}$ found in the previous section.

$$\sigma_r = \frac{\sigma_o}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_o}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{\sigma_o}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_o}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{\sigma_o}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

The plots of the normal stress distribution along the transverse and centerline of the infinitely large plate is shown in Fig. 3.15.

It can be seen that the normal stress occurred on the edge of the hole where $r = a$ and $\theta = \pm\dfrac{\pi}{2}$ has the positive value of

$$(\sigma_\theta)_{\max} = 3\sigma_o$$

For $\theta = 0$ and $\theta = \pi$, the normal stress on the edge of the hole where $r = a$ has the negative value of σ_o. Thus, σ_θ attains a maximum tensile value of three times the uniformly distributed stress σ_o. This value is the largest stress occurs in the plate. Hence the stress concentration factor at the hole, which is the ratio of the maximum normal stress at the hole divided by the averaged normal stress at the same point in the absence of the hole, is 3

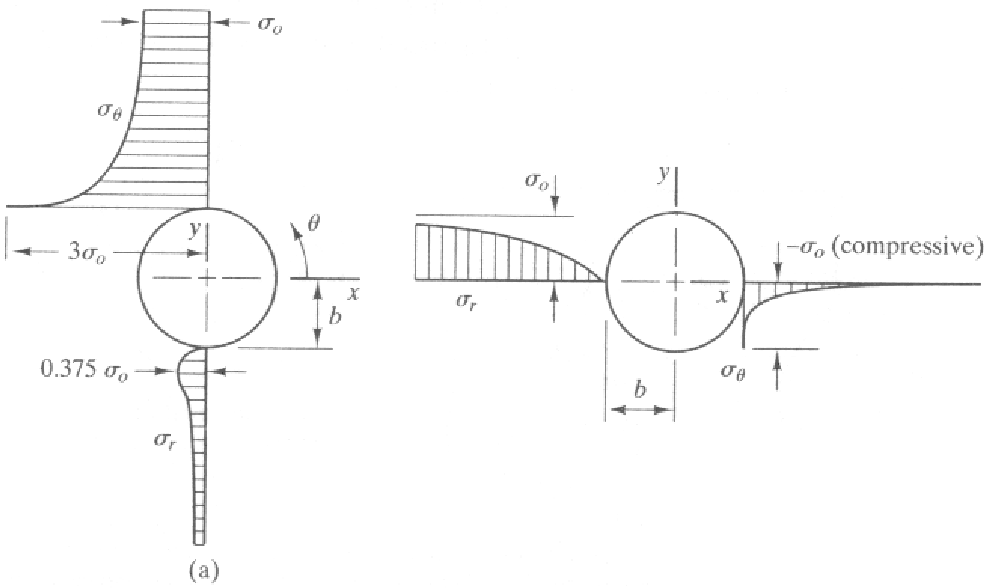


Fig. 3.15

From Fig. 3.15, it can be seen that the stress σ_θ approaches the average values of σ_o at a small distance from the hole. Thus, the high stress gradient or stress concentration is quite localized in effect according to the Saint-Venant's principle.

The state of stresses for a circular hole in a plate under other states of plane stress can be determined by using the principle of superposition. For example, if the previously obtained the state of stresses is combined with another state of stresses in which everything is rotated by 90^o, we obtain the state of stresses in equal biaxial tension. Or, if the direction of σ_o is reversed in one of these two solutions, we obtain results for the pure shear.

It should be noted that the obtained stress equations can only be used in the case when the plate has the diameter that is small compared to the width of the plate. When the diameter of the hole is comparable to the width of the plate, researches have shown that the maximum normal stress should be calculated by using the equation

$$\sigma_{\max} = \frac{3\kappa - 1}{\kappa + 0.3}\sigma_n$$

where κ is the ratio of the width of the plate to the diameter of the hole, and σ_n is the averaged stress over the cross-sectional area at the hole.

3.10 Concentrated Force at a Point of a Straight Boundary

Consider a concentrated vertical force P acting on horizontal straight boundary of an infinitely large plate as shown in Fig. 3.16a. The concentrated force P is considered as a uniformly distributed force along the thickness-direction line. The thickness of the plate is taken as unity. Thus, P has the unit of force per unit thickness.

Force on a Straight Edge

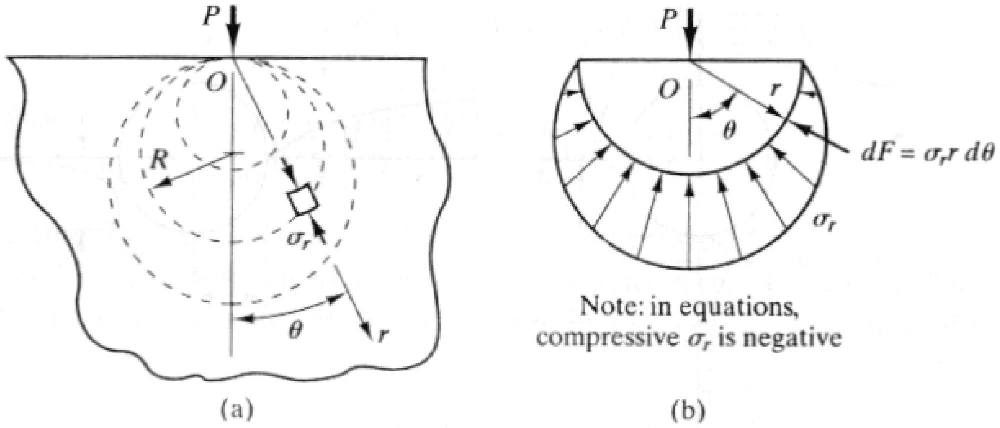


Fig. 3.16

The resulting stresses due to the concentrated force P can be determined by using the Airy stress function of the form

$$F = a_1 r\theta \sin\theta$$

The corresponding stresses can be determined as

$$\sigma_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} = 2a_1\frac{\cos\theta}{r}$$

$$\sigma_\theta = \frac{\partial^2 F}{\partial r^2} = 0$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right) = 0$$

The actual stress distribution of σ_r is shown in Fig 3.16b. The constant a_1 can be determined by using the summation of the vertical forces to be zero.

$$-\int_{-\pi/2}^{\pi/2}(\cos\theta)\sigma_r r d\theta - P = 0$$

$$a_1 = -\frac{P}{\pi}$$

Thus, the state of stresses is

$$\sigma_r = -\frac{2P\cos\theta}{\pi r}$$

$$\sigma_\theta = 0 \text{ and } \tau_{r\theta} = 0.$$

This distribution of stress is called a simple radial distribution, every polar coordinate element at a distance r from the point of application of P, being in simple compression in the radial direction. The stress σ_r becomes very large as r becomes small and is undefined for $r = 0$ (the stress is said to be singular). For all real materials, yielding will occur in the neighborhood of the load, resulting in a plastic zone.

If it is assumed that the plastic zone is sufficiently small so that the presence of the yielding material does not significantly change the elastic solution,

$$\frac{r}{\cos\theta} = \frac{2P}{\pi\sigma_y} = d_y = \text{constant}$$

Thus, under the load, there is a circular plastic zone of diameter d_y.

In general, there exists a family of circles of diameter $d = r/\cos\theta$ as shown by the dashed line in Fig. 3.16a that σ_r is constant. This circle line is called circle of constant stress σ_r.

Chapter 4

Applications of Energy Method

4.1 Degree of Freedom

Degree of freedom is the independent quantities used to define a configuration of a system that violates neither compatibility conditions nor constraints.

Finite degree of freedom system

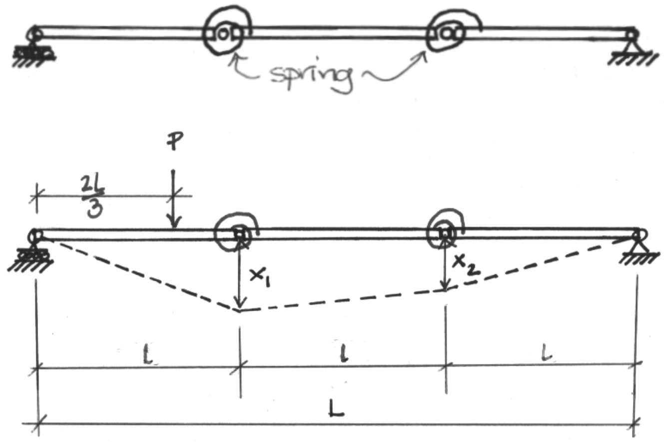


Fig. 4.1

Consider the beam having three **rigid** elements connected with hinges and elastic springs as shown in Fig. 4.1. In this case, the configuration of the beam can be described by using two of the independent quantities x_i and θ_j where $i \neq j$ such as x_1 and x_2 or x_1 and θ_2. Thus, the beam has 2 degree of freedom.

Infinite degree of freedom system

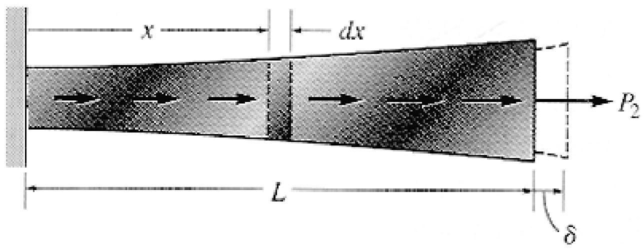


Fig. 4.2

Consider the bar subjected to arbitrary axial load as shown in Fig. 4.2. It requires an infinite number of degree of freedom to describe its axial displacement since the displacement varies along the length of the bar.

$$\sigma = \frac{P(x)}{A(x)} \qquad\qquad \varepsilon = \frac{d\delta}{dx}$$

$$\sigma = E\varepsilon$$

$$\frac{P(x)}{A(x)} = E\frac{d\delta}{dx}$$

$$d\delta = \frac{P(x)\,dx}{A(x)\,E}$$

$$\delta = \int_0^L \frac{P(x)}{A(x)\,E}dx$$

However, the displacement can be idealized or approximated as

$$\delta = u = a_1 x + a_2 x^2 + \cdots + a_n x^n$$

which describe the axial displacement of the bar with n degree of freedom and a_i is usually called *generalized coordinate*.

4.2 Work and Energy

In every type of system, forces are present to which may be associated a *capacity* to displace and, thus, capacity to perform work. It is indicative of the *energy possessed* by the system. Thus, energy is the *capacity* to do work.

Force in a system may *perform* work, but the system *possesses* energy. In order to evaluate the amount of work done in a physical process, we need to know only the *change* in energy. Thus, *the reference or datum with respect to which we measure is completely arbitrary*. Consider, for example, the work done by the force f in bringing a particle from point A to point B as shown in Fig 4.3.

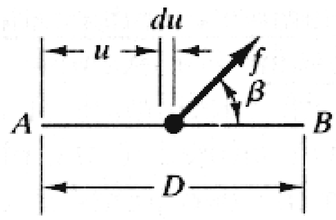


Fig. 4.3

$$dW = f\cos\beta(du)$$

$$W = \int_A^B f\cos\beta du$$

If the angle β is $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, the force f performs positive work. If the angle β is $\frac{\pi}{2} < \beta < \frac{3\pi}{2}$, the force f performs negative work. The $f\sin\beta$ does no work. If the system is conservative, the work done is independent of the path from point A to point B. Then, if the integral is independent of the path, the quantity $(f\cos\beta)du$ is an exact differential of some function V. Thus, for conservative system,

$$W = \int_A^B f \cos \beta du = \int_A^B dV = V_B - V_A = -\Delta V$$

where ΔV is the change in V from point A to point B.

The function V is called *potential function*, or, in this discussion, the *potential energy* of the system. Physically, the potential energy is the capacity of a conservative system to perform work by virtue of its configuration with respect to an arbitrary datum. Also, the change in potential energy is a negative of the work done. For example, if we lift a weigh mg to a height of h from a reference plane, the work done we perform is

$$W = -mgh$$

The negative sign indicates the opposite directions of the weight (gravitational force) and the vertical movement. According to the law of conservation of energy, the potential energy of the mass m is increased by an amount of mgh.

$$V = mgh$$

Potential energy in structural system, V

For the conservative of force, the work to move a mass does not depend upon the route of moving, but does depend upon the starting point and terminating point. A conservative system has a total potential energy. We can express the energy content of the system in terms of its configuration, without reference to whatever deformation history may have led to that configuration.

The total potential energy includes:

1. the potential of the external forces to do work, Ω
2. the internal strain energy of elastic distortion, U

$$V = \Omega + U$$

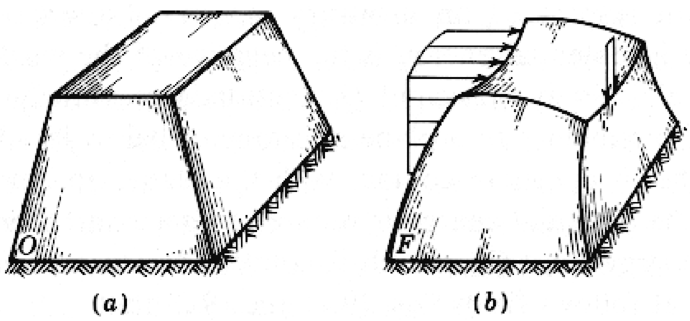


Fig. 4.4

Consider a three-dimensional deformable body as shown in Fig. 4.4a which is in a state of undeformed configuration and the external forces have zero potential energy. As the body slowly deforms, external forces move through true displacements and perform *external* *work* until the final configuration is reached as shown in Fig. 4.4b. Again, let us assume that the work done by the external forces in deforming the body is independent of the path from Fig. 4.4a to Fig. 4.4b. Hence, if W_e is the total work done by the external forces, dW_e is the exact differential of a potential function Ω which is the potential of the external forces to do work. The change in Ω from the configuration in Fig. 4.4a to Fig. 4.4b is $-W_e$. Therefore, from the definition of work,

$$\Omega = -\iiint_V \left[\int_a^b (B_x du + B_y dv + B_z dw) \right] dV - \iint_{S_1} \left[\int_a^b (X_s du + Y_s dv + Z_s dw) \right] dS$$

where integral inside the bracket are carried from the initial state to the final deformed state as shown in Fig. 4.4a to Fig. 4.4b, respectively. The S_1 is the portion of the total body surface area subjected to the surface forces. Since the external forces are independent of the displacements,

$$\Omega = -\iiint_V (B_x u + B_y v + B_z w) dV - \iint_{S_1} (X_s u + Y_s v + Z_s W) dS$$

If the body forces are negligible and the surface forces are represented by a system of concentrated forces and moments P_1, P_2, …, P_n with the corresponding displacement of Δ_1, Δ_2, …, Δ_3,

$$\Omega = -(P_1\Delta_1 + P_2\Delta_2 + \ldots + P_n\Delta_n)$$

The internal forces developed in a deformable body also posses a capacity to perform work. Under the action of external forces, the body is deformed and the stresses are developed which results in the internal forces. The internal forces perform work while the body are deformed. If the strained body is allowed to slowly return to its unstrained state, it will return the work done by the external forces. This capacity of internal forces to perform work as the body returns to the unstrained state is called *strain energy*.

For linearly elastic and isotropic material, the strain energy density was derived as

$$U_o = \frac{1}{2}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}\right]$$

The total strain energy U of the body is

$$U = \iiint_V U_o dV$$

4.3 Principle of Stationary Potential Energy

Consider an element of unit volume in a one-dimensional body subjected only to the stress and strain components σ_x and ε_x. By the definition of virtual work, the internal virtual work due to a virtual strain is

$$\delta U = \sigma_x \delta \varepsilon_x$$

From calculus, the differential change in the strain energy in the body due to an increment $d\varepsilon_x$ of the strain is

$$dU = \frac{\partial U}{\partial \varepsilon_x} d\varepsilon_x = \sigma_x d\varepsilon_x$$

We can see that the symbol δ is not just only the symbol for the virtual quantity. In fact, it behaves as a *variational operator* which obeys the rules of operation similar to those of the first differential operator d. In analogy, if we refer δ as *first variation*, we can see that the internal virtual work can be interpreted as the first variation in the strain energy due to variations in the components of strains.

Similarly, if W_e is the work done by external forces in a conservative system

$$\Omega = -W_e$$

$$\delta\Omega = -\delta W_e$$

From the principle of virtual displacements $\delta W_e = \delta U$,

$$\delta U = -\delta\Omega$$

$$\delta U + \delta\Omega = 0$$

$$\delta(U + \Omega) = 0$$

$$\delta V = 0$$

A deformable body is in equilibrium if the first variation in the potential energy of the system is zero for every virtual displacement consistent with the constraints

Consider a system having two degrees of freedoms x_1 and x_2 as shown in Fig. 4.6. In this case, the total potential energy of the system can be expressed as a function of x_1 and x_2, so that are the rate of change of V with respect to x_1 and x_2. Thus, for the structure in equilibrium, if first x_1 is given a variation δx_1 and then x_2 a variation δx_2,

$$\delta V = \frac{\partial V}{\partial x_1} \delta x_1 = 0$$

$$\delta V = \frac{\partial V}{\partial x_2} \delta x_2 = 0$$

Since the magnitude of the virtual displacement δx_1 and δx_2 are arbitrary,

$$\frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial x_2} = 0$$

which mean for equilibrium to exist,

$$dV = \frac{\partial V}{\partial x_1} dx_1 + \frac{\partial V}{\partial x_2} dx_2 = 0$$

Recall from calculus, the total differential of a function vanishes at the *critical points* of the function at which the function is a relative maximum or maximum or they may be saddle points at which the function is minimax. At such point, the function is said to assume a *stationary value*. Since the equation $dV = \frac{\partial V}{\partial x_1} dx_1 + \frac{\partial V}{\partial x_2} dx_2 = 0$ is valid only at the equilibrium, thus, the *principle of stationary potential energy* can be stated that

If a structure is in static equilibrium, the total potential energy of the system has a stationary value.

The equation $dV = \frac{\partial V}{\partial x_1} dx_1 + \frac{\partial V}{\partial x_2} dx_2 = 0$ can be interpreted another way. Let V is a continuous function displacement, which is consistent with the boundary conditions. dV is zero only if the displacements are corresponding to the equilibrium configurations. Thus, the *principle of stationary potential energy* can be restated that

Of all the possible displacements which satisfy the boundary conditions of a structural system, those corresponding to the equilibrium configurations make the total potential energy assume a stationary value.

Example 4-1

Determine the equilibrium configurations of a system of three bars subjected to the point load P as shown in Fig. Ex 4-1a. The bars are supported by pins and are joined with internal hinge and springs having the stiffness of $k = \dfrac{EI}{l}$.

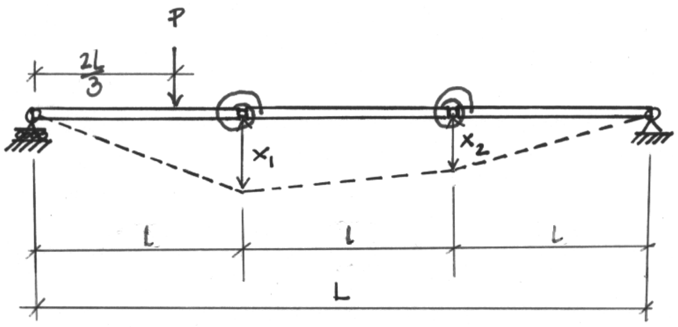


Fig. Ex 4-1a

The system as shown in Fig. Ex 4-1a has two degrees of freedoms x_1 and x_2.

The potential energy of the external force P is equal to the amount of work done by the force P referred to the reference at support.

$$\Omega = -P\left[\frac{2}{3}x_1\right]$$

(Negative value of the work done by external force = the external potential energy)

The work done due to the couple moment M_o using to move the spring from 0 to ϕ as shown in Fig. Ex 4-1b or the strain energy stored in the spring is equal to

$$\int_0^{\phi} M_o d\theta = \int_0^{\phi} k\theta\, d\theta = \frac{1}{2}k\phi^2 .$$

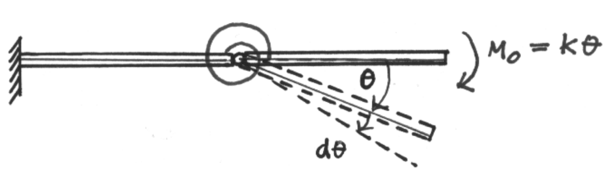


Fig. Ex 4-1b

For small displacement and rotation, the angle

$$\theta \approx \sin\theta \approx \tan\theta .$$

Then, from Fig. Ex 4-1c,

$$\theta_1 = \frac{x_1}{l} + \frac{x_1 - x_2}{l} = \frac{2x_1}{l} - \frac{x_2}{l}$$

$$\theta_2 = \frac{x_2}{l} - \frac{x_1 - x_2}{l} = \frac{2x_2}{l} - \frac{x_1}{l}$$

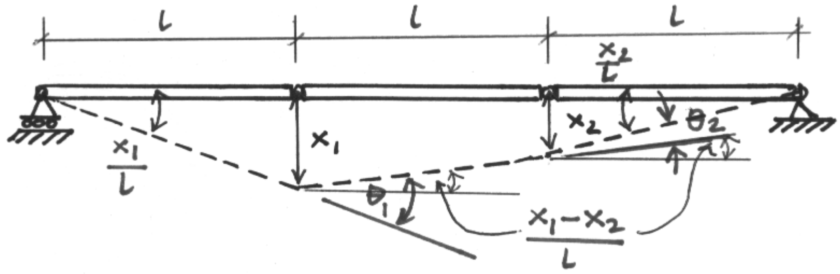


Fig. Ex 4-1c

Therefore, the strain energy stored in the springs on the beam is

$$U = \frac{1}{2}\frac{EI}{l}\left(\frac{2x_1}{l} - \frac{x_2}{l}\right)^2 + \frac{1}{2}\frac{EI}{l}\left(\frac{2x_2}{l} - \frac{x_1}{l}\right)^2$$

The total potential energy is

$$V = -P\left[\frac{2}{3}x_1\right] + \frac{1}{2}\frac{EI}{l}\left(\frac{2x_1}{l} - \frac{x_2}{l}\right)^2 + \frac{1}{2}\frac{EI}{l}\left(\frac{2x_2}{l} - \frac{x_1}{l}\right)^2$$

For the stable equilibrium system, the total potential energy must be minimum.

$$\frac{\partial V}{\partial x_1} = 0$$

$$\frac{\partial V}{\partial x_2} = 0$$

Solving the simultaneous equation, we have $x_1 = \frac{5x_2}{4}$. Then,

$$x_1 = \frac{10}{27}\frac{Pl^3}{EI}$$

$$x_2 = \frac{8}{27}\frac{Pl^3}{EI}$$

Example 4-2

The three bars plane truss has the configuration as shown in Fig. Ex 4-2. If the bars have the same EA, determine the stress in each bar.

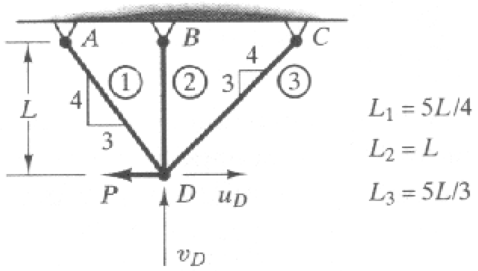


Fig. Ex 4-2

Let u_D and v_D are the degree of freedom at joint D. By considering the deformation of each bar, we can find the relationships of the change in length of each bar δ_i and the degree of freedom u_D and v_D in the form of

$$\delta_1 = \frac{3}{5}u_D - \frac{4}{5}v_D$$

$$\delta_2 = -v_D$$

$$\delta_3 = -\frac{4}{5}u_D - \frac{3}{5}v_D$$

The axial strain in each bar is

$$\varepsilon_1 = \frac{\delta_1}{L_1} = \frac{1}{L_1}\left[\frac{3}{5}u_D - \frac{4}{5}v_D\right] = \frac{1}{L}\left[\frac{12}{25}u_D - \frac{16}{25}v_D\right]$$

$$\varepsilon_2 = \frac{\delta_2}{L_2} = -\frac{v_D}{L}$$

$$\varepsilon_3 = \frac{\delta_3}{L_3} = \frac{1}{L_3}\left[-\frac{4}{5}u_D - \frac{3}{5}v_D\right] = \frac{1}{L}\left[-\frac{12}{25}u_D - \frac{9}{25}v_D\right]$$

Strain energy in a uniform bar is $U_a = \int_0^L \frac{P^2}{2AE}dx$. From the Hooke's law, $P = AE\varepsilon$.

Thus,

$$U_a = \int_0^L \frac{AE\varepsilon^2}{2}dx = \frac{AEL}{2}\varepsilon^2$$

The total strain energy is

$$U = \sum_{i=1}^{3}\frac{A_iE_iL_i}{2}\varepsilon_i^2 = \frac{AE}{2}\left[\sum_{i=1}^{3}L_i\varepsilon_i^2\right] = \frac{AE}{2}\left[L_1\varepsilon_1^2 + L_2\varepsilon_2^2 + L_3\varepsilon_3^2\right]$$

The external potential energy is equal to negative of the external work.

$$\Omega = Pu_D$$

Thus, the total potential energy of the system is

$$V = \frac{AE}{2}\left[L_1\varepsilon_1^2 + L_2\varepsilon_2^2 + L_3\varepsilon_3^2\right] + Pu_D$$

$$V = \frac{AE}{2}\left[\frac{5}{4L}\left[\frac{12}{25}u_D - \frac{16}{25}v_D\right]^2 + \frac{v_D^2}{L} + \frac{5}{3L}\left[-\frac{12}{25}u_D - \frac{9}{25}v_D\right]^2\right] + Pu_D$$

From the principle of stationary potential energy

$$\frac{\partial V}{\partial u_D} = 0$$

$$\frac{\partial V}{\partial v_D} = 0$$

Solving the simultaneous equations for u_D and v_D.

Substituting u_D and v_D into the strain equations, we get the axial strain in each bar.

Finally, the stress in each bar can be determined by using the Hooke's law.

4.4 Principle of Minimum Potential Energy

An equilibrium configuration is stable if the system return to its original configuration after being given a small disturbance.

Consider pendulum of weight W in Fig. 4.5. The position A and B are both equilibrium configurations since there is a balance of forces. The configuration A is one of unstable equilibrium since during a small angular displacement, positive work is done, the potential energy decreases and the kinetic energy increase significantly. The configuration B is one of stable equilibrium since during a small angular displacement, negative work is done, the potential energy increases and create only an infinitesimal change in kinetic.

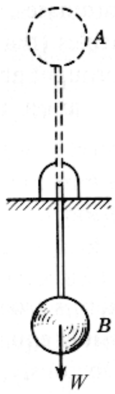


Fig. 4.5

If small displacement cause no change in potential energy, the system is said to be in neutral equilibrium, which is also the unstable equilibrium.

Consider the motion of a rigid marble having a weight of W along a smooth contour in Fig 4.6. The rigid marble has no strain energy.

$$U = 0$$

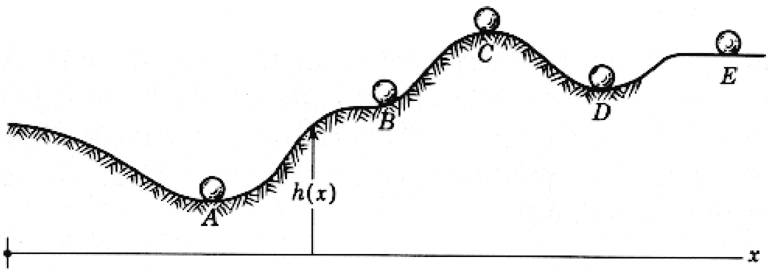


Fig 4.6

If x-axis is the reference line, the total potential of a given marble is

$$V = \Omega = Wh(x)$$

Generalized coordinate is the least number of real independent variables required to specify the configuration of a system. The rigid bars as shown in Fig. Ex 4-1 require two generalized coordinates to define their configuration. The virtual displacements in a system, which are consistent with the constraints, can be expressed as a function of the variations in the generalized coordinates

A single marble in Fig. 4.6 can be specified by the generalized coordinate x. Thus, a virtual displacement following the rigid contour of a marble and having the generalized coordinate x is a small variation δx. The variation δx produces a variation in V. Since each marble is in equilibrium,

$$\delta V = W\frac{dh}{dx}\delta x = 0$$

which is valid when $\dfrac{dh}{dx} = 0$ at points A, B, C, D, and E. Thus, these points are possible equilibrium configurations.

From the previous discussion, it may be concluded that

1. The marbles at A and D are in stable equilibrium.
2. The marble at C is in unstable equilibrium.
3. The marble at E is in neutral equilibrium.

The potential energy of the marble at point B is either increased or decreased depending on the direction of the virtual displacement. Thus, the marble at B is in unstable equilibrium.

If $V(x)$ corresponds to an equilibrium configuration, $V(x+\delta x)$ is the potential energy of a configuration in the neighborhood of $V(x)$ if δx is sufficiently small. By using Taylor's series, we have

$$V(x+\delta x) = V(x) + \frac{dV(x)}{dx}\delta x + \frac{1}{2}\frac{d^2V(x)}{dx^2}(\delta x)^2 + \frac{1}{3}\frac{d^3V(x)}{dx^3}(\delta x)^3 + \ldots$$

For equilibrium, $\dfrac{dV}{dx} = 0$. Thus, the change in V due to δx is $V(x+\delta x) - V(x)$,

$$\Delta V = \frac{1}{2}\frac{d^2V(x)}{dx^2}(\delta x)^2 + \frac{1}{3}\frac{d^3V(x)}{dx^3}(\delta x)^3 + \ldots$$

For the system in Fig. 4.6,

$$\Delta V = W\left[\frac{1}{2}\frac{d^2h}{dx^2}(\delta x)^2 + \frac{1}{3}\frac{d^3h}{dx^3}(\delta x)^3 + \ldots\right]$$

Thus, the first order variation in the displacements at an equilibrium configuration causes a change in potential energy of order $(\delta x)^2$.

If the term $\dfrac{d^2V}{dx^2}$ of ΔV is not zero, the sign of ΔV is independent of the sign of the virtual displacement δx since $(\delta x)^2$. Then, V is either a relative maximum or relative minimum. For the system in Fig. 4.6, the term $\frac{d^2h}{dx^2}$, representing the curvature of the rigid curve, is positive at A and D and negative at C. Thus, V is a relative minimum at A and D and a relative maximum at C. Clearly, the marbles at A and D are in stable equilibrium and the marble at C is in unstable equilibrium. At B, $\frac{d^2h}{dx^2} = 0$ due to flat contour at the point. However, if $\frac{d^3h}{dx^3} \neq 0$, the sign of ΔV depends on the sign of the virtual displacement δx due to the term $(\delta x)^3$. At this point, V is a minimax and the marble is in unstable equilibrium. At E, all derivative of V vanish and ΔV is zero for small virtual displacement. Thus, The marble at E is in neutral equilibrium.

From these observations, we conclude that

1. If $\delta^2 V > 0$, the system is in stable equilibrium.
2. If $\delta^2 V < 0$, the system is in unstable equilibrium.
3. If $\delta^2 V = 0$, the system is in neutral equilibrium.

Example 4-3

Check the stability of the system of rigid bar subjected to axial load P as shown in Fig. Ex 4-3.

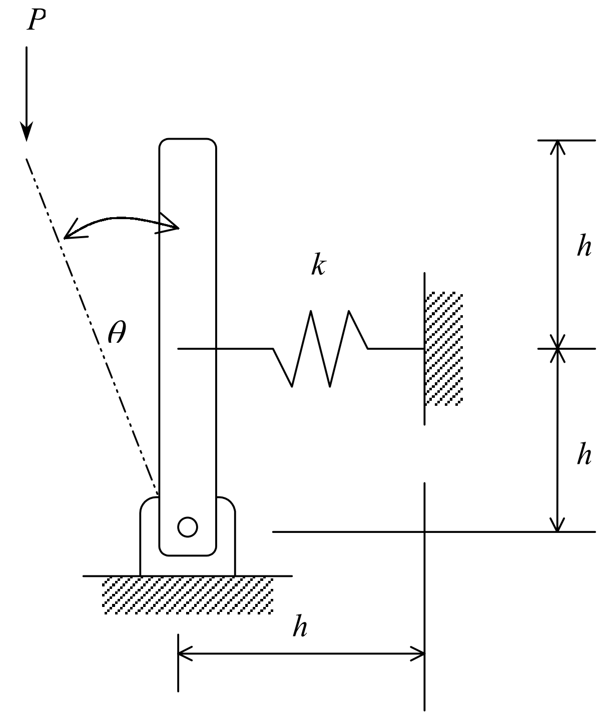


Fig. Ex 4-3

Consider a system of rigid bar subjected to axial load P as shown in Fig. Ex 4-3. All configurations of the system can be described by specifying one independent variable, the angular coordinate θ. Thus, θ is the generalized coordinate.

The external potential energy Ω of the system is zero for reaction forces plus minus $2Ph(1-\cos\theta)$ due to axial force P

$$\Omega = -2Ph(1-\cos\theta)$$

The internal potential energy or strain energy U of the system is zero for rigid bar plus $\frac{1}{2}k(h\sin\theta)^2$ in the spring.

$$U = \frac{1}{2}k(h\sin\theta)^2$$

The total potential energy of the system is

$$V(\theta) = \frac{1}{2}k(h\sin\theta)^2 - 2Ph(1-\cos\theta)$$

The total potential energy of the system due to variations $\delta\theta$ is

$$V(\theta+\delta\theta) = V(\theta) + \frac{dV(\theta)}{d\theta}\delta\theta + \frac{1}{2}\frac{d^2V(\theta)}{d\theta^2}(\delta\theta)^2 + \frac{1}{3}\frac{d^3V(\theta)}{d\theta^3}(\delta\theta)^3 + \ldots$$

The change in the total potential energy of the system due to a rotation variation $\delta\theta$ is

$$\Delta V = V(\theta+\delta\theta) - V(\theta) = \frac{dV}{d\theta}\delta\theta + \frac{1}{2}\frac{d^2V}{d\theta^2}(\delta\theta)^2 + \frac{1}{3}\frac{d^3V}{d\theta^3}(\delta\theta)^3 + \dots$$

$$\Delta V = \left[kh^2 \sin\theta\cos\theta - 2Ph\sin\theta\right]\delta\theta + \frac{1}{2}\left[kh^2\cos 2\theta - 2Ph\cos\theta\right](\delta\theta)^2 + \dots$$

For equilibrium configurations, $\delta V = 0$.

$$\frac{dV}{dx} = \sin\theta\left[kh^2\cos\theta - 2Ph\right] = 0$$

This equation is satisfied when $\sin\theta = 0$ or $kh^2\cos\theta - 2Ph = 0$.

If $\sin\theta = 0$,

$$\theta = 0^o \text{ or } \theta = 180^o.$$

If $kh^2\cos\theta - 2Ph = 0$,

$$\cos\theta = \frac{2P}{kh}$$

This means that θ can be any values in the first or fourth quadrants of the reference coordinate. This solution is trivial.

For stable equilibrium configurations, $\delta^2 V > 0$.

If $\theta = 0^o$, the term

$$\frac{d^2V}{dx^2} = kh^2\cos 2\theta - 2Ph\cos\theta = kh^2 - 2Ph$$

which mean that, **for $\theta = 0^o$, the system will be in stable equilibrium only if $kh > 2P$**. Then, the critical load can be determined by setting $kh = 2P$.

$$P_{cr} = \frac{kh}{2}$$

It should be noted that, for small rotation θ, this critical load can also be found by using the mechanics of material by using the equilibrium of the moment about the pinned support.

$$P_{cr}(2h\theta) - kh^2\theta = 0$$

$$P_{cr} = \frac{kh}{2}$$

If $\theta = 180^o$, the term

$$kh^2\cos 2\theta - 2Ph\cos\theta = kh^2 + 2Ph$$

which is always larger than zero. Thus, **the system is always in stable equilibrium when $\theta = 180^o$.**

It should be noted that in this case, we can not assume $\cos\theta = 1 - \frac{1}{2}\theta^2$ in the small displacement analysis.

$$V = \frac{1}{2}k(h\theta)^2 - 2Ph\left[1-(1-\frac{1}{2}\theta^2)\right] = \frac{1}{2}kh^2\theta^2 - Ph\theta^2$$

$$\Delta V = \left[kh^2\theta - 2Ph\theta\right]\delta\theta + \frac{1}{2}\left[kh^2 - 2Ph\right](\delta\theta)^2 + \ldots$$

For equilibrium, $\delta V = 0$, we have

$$\frac{dV}{d\theta} = \theta\left[kh^2 - 2Ph\right] = 0$$

which provides only one solution that is $\theta = 0^o$. Thus, for stability analysis, we have to consider the potential energy of the system to the cubic terms.

4.5 Second Variation in the Total Potential Energy

The change in the potential energy of a system can be expressed as

$$\Delta V = \delta V + \frac{1}{2}\delta^2 V + \text{terms of higher order}$$

where $\delta^2 V$ is the second variation in V in analogy with the first variation δV

$$\delta^2 V = \frac{d^2 V}{dx^2}(\delta x)^2$$

Since in equilibrium conditions δV is zero, the sign of ΔV is often determined by the sign of $\delta^2 V$. Thus, the second variation often plays an important role in the study of the stability of the structural systems.

The second variation term is quadratic in δx since it is a function of $(\delta x)^2$. Thus, if V is a function of two generalized coordinates x and y and if the point (a , b) correspond to an equilibrium configuration, by using Taylor's series expansion, we have

$$\begin{aligned}\Delta V &= V(a+\delta x, b+\delta y) - V(a,b) \\ &= \frac{\partial V(a,b)}{\partial x}\delta x + \frac{\partial V(a,b)}{\partial y}\delta y + \frac{1}{2}\left[\frac{\partial^2 V(a,b)}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 V(a,b)}{\partial x \partial y}\delta x \delta y + \frac{\partial^2 V(a,b)}{\partial y^2}(\delta y)^2\right] \\ &\quad + \text{ terms of higher order}\end{aligned}$$

For equilibrium,

$$\delta V = \frac{\partial V(a,b)}{\partial x}\delta x + \frac{\partial V(a,b)}{\partial y}\delta y = 0$$

For stable equilibrium,

$$\begin{aligned}\delta^2 V &= \frac{\partial^2 V(a,b)}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 V(a,b)}{\partial x \partial y}\delta x \delta y + \frac{\partial^2 V(a,b)}{\partial y^2}(\delta y)^2 \\ &= a_{11}(\delta x)^2 + a_{12}\delta x \delta y + a_{21}\delta x \delta + a_{22}(\delta y)^2\end{aligned}$$

This equation is a quadratic form in δx and δy. It should be noted that $a_{12} = a_{21}$. Clearly, the sign of ΔV depends on the sign of the coefficients a_{11}, a_{12}, a_{21}, and a_{22} which are the functions of the applied loads. In general, there are five types of quadratic forms:

1. Positive definite.
2. Positive semidefinite.
3. Negative definite.
4. Negative semidefinite.
5. Indefinite.

If $\delta^2 V$ is a positive definite quadratic form, it can be shown that ΔV is positive. Thus, V is a relative minimum. If $\delta^2 V$ is negative definite, negative semidefinite, or indefinite, ΔV is negative and the equilibrium is unstable. If $\delta^2 V$ is positive semidefinite or zero, higher variation must be considered.

In the stability analysis of linearly elastic system with small displacements systems, V can be expressed as a quadratic function. When such systems are stable, $\delta^2 V$ is positive definite and the determinant

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$$

If the applied loads are increased, D decreases. When a critical load is reached, $\delta^2 V$ is not a positive definite and the system becomes unstable and buckles. Thus, the stability criterion is

$$\delta^2 V = D = 0$$

Example 4-4: Stability analysis of simple structures

Determine the stability configurations of the system of rigid bars subjected to concentrically axial load P as shown in Fig. Ex 4-4a.

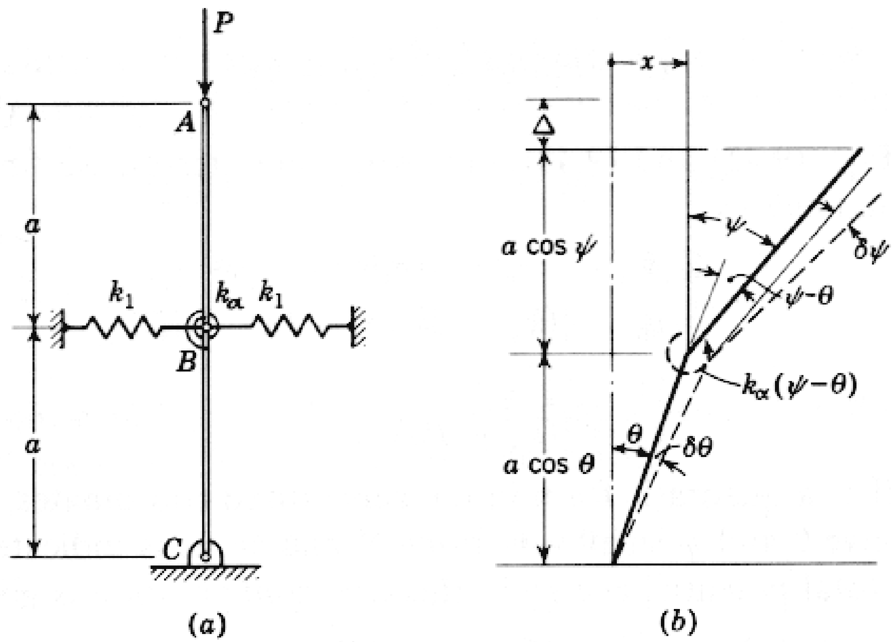


Fig. Ex 4-4

The system of rigid bars as shown in Fig. Ex 4-4a has two degree of freedoms. All configurations of the system can be described by specifying two independent variables, the angular coordinate θ and ψ. Thus, θ and ψ are the generalized coordinates.

For positive values of θ and ψ, the applied load P is displaced a distance Δ.

$$\Delta = 2a - a\cos\theta - a\cos\psi$$

Thus, the potential energy of the external force P is

$$\Omega = -P\Delta$$

For small values of generalized coordinates θ and ψ and since we are interested to find only the critical load, thus, we let $\cos\theta = 1 - \frac{1}{2}\theta^2$. The potential energy of the external force P is

$$\Omega = -\frac{Pa}{2}(\theta^2 + \psi^2)$$

Since the internal forces and moments are developed in the springs, the system posses strain energy.

$$U = 2\int_0^x k_1 x dx + \int_0^\alpha k_\alpha \alpha d\alpha$$

$$U = 2\left(\frac{k_1 x^2}{2}\right) + \frac{k_\alpha \alpha^2}{2}$$

where x is the change in length of the linear spring and α is the angle of rotation of the rotational spring.

From the geometry of Fig. Ex 4-4b, we find

$$x = a\sin\theta \approx a\theta$$

$$\alpha = \psi - \theta$$

Thus,

$$U = k_1 a^2 \theta^2 + \frac{1}{2} k_\alpha (\psi - \theta)^2$$

The total potential energy of the system is

$$\begin{aligned} V(\theta,\psi) &= \frac{1}{2}(k_\alpha + 2k_1 a^2 - Pa)\theta^2 - k_\alpha \theta\psi + \frac{1}{2}(k_\alpha - Pa)\psi^2 \\ &= c_0\theta^2 + c_1\theta\psi + c_2\psi^2 \end{aligned}$$

where

$$c_0 = \frac{1}{2}(k_\alpha + 2k_1 a^2 - Pa)$$

$$c_1 = -k_\alpha$$

$$c_2 = \frac{1}{2}(k_\alpha - Pa)$$

The total potential energy of the system due to variations $\delta\theta$ and $\delta\psi$ is

$$\begin{aligned} V(\theta + \delta\theta, \psi + \delta\psi) = c_0\theta^2 + c_1\theta\psi + c_2\psi^2 \\ + \left[(2c_0\theta + c_1\psi)\delta\theta + (c_1\theta + 2c_2\psi)\delta\psi\right] \\ + \left[c_0(\delta\theta)^2 + c_1\delta\theta\delta\psi + c_2(\delta\psi)^2\right] \end{aligned}$$

The change in total potential energy of the system is

$$\Delta V = \left[(2c_0\theta + c_1\psi)\delta\theta + (c_1\theta + 2c_2\psi)\delta\psi\right] + \left[c_0(\delta\theta)^2 + c_1\delta\theta\delta\psi + c_2(\delta\psi)^2\right]$$

For equilibrium configurations, δV is zero.

$$\frac{\partial V}{\partial \theta} = 0 = 2c_0\theta + c_1\psi$$

$$\frac{\partial V}{\partial \psi} = 0 = c_1\theta + 2c_2\psi$$

Clearly, $\theta = \psi = 0$ is an equilibrium configuration.

For stable equilibrium configurations, $\delta^2 V$ is greater than zero.

$$\delta^2 V = \frac{\partial^2 V}{\partial \theta^2}(\delta\theta)^2 + 2\frac{\partial^2 V}{\partial\theta\partial\psi}\delta\theta\delta\psi + \frac{\partial^2 V}{\partial\psi^2}(\delta y)^2$$
$$= 2\left[c_0(\delta\theta)^2 + c_1\delta\theta\delta\psi + c_2(\delta\psi)^2\right]$$

When the system is stable, the determinant

$$D = \begin{vmatrix} c_0 & c_1/2 \\ c_1/2 & c_2 \end{vmatrix} > 0$$

When P reaches the critical load, $D = 0$ and the system becomes unstable. Thus,

$$c_0 c_2 - \frac{1}{4}c_1^2 = 0$$

Substituting $c_0 = \frac{1}{2}(k_\alpha + 2k_1 a^2 - Pa)$, $c_1 = -k_\alpha$, and $c_2 = \frac{1}{2}(k_\alpha - Pa)$, we have

$$(k_\alpha - Pa)\ (k_\alpha + 2k_1 a^2 - Pa) - k_\alpha^2 = 0$$

Solving this polynomial equation for P, we have

$$P_1 = \frac{1}{a}\left[k_\alpha + k_1 a^2 - \sqrt{k_\alpha^2 + k_1^2 a^4}\right]$$

$$P_2 = \frac{1}{a}\left[k_\alpha + k_1 a^2 + \sqrt{k_\alpha^2 + k_1^2 a^4}\right]$$

Let $k_\alpha = k_1 a^2 = k$, then,

$$P_1 = 0.586\frac{k}{a}$$

and

$$P_2 = 3.414\frac{k}{a}$$

Thus, $P_1 = 0.586\frac{k}{a}$ is the critical load for the system. Substituting $P_{cr} = 0.586k/a$ into the equations $2c_0\theta + c_1\psi = 0$ or $c_1\theta + 2c_2\psi = 0$, we find

$$\psi = 2.414\theta$$

The buckled shape for this case is shown in Fig. Ex 4-4c(a).

Substituting $P_{cr} = 3.414k/a$ into the equations $2c_0\theta + c_1\psi = 0$ or $c_1\theta + 2c_2\psi = 0$, we find

$$\psi = -0.414\theta$$

The buckled shape for this case is shown in Fig. Ex 4-4c(b). These two possible buckled shapes are called the *buckling modes* of the system.

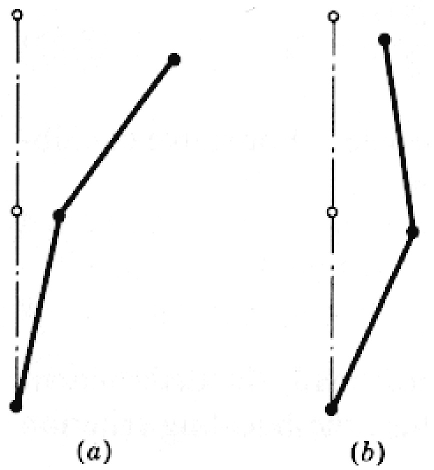


Fig. Ex 4-4c

Lagrange multipliers

In many cases when it is not convenient to express V in terms of the least number of independent variables. Then, the problems become one of minimizing a function whose variables are constrained by some side relationships. These kinds of problems can be solved by using the ***Lagrange multipliers***.

In general, when the variables x_1, x_2, …, x_n of a function $G(x_1, x_2, \ldots, x_n)$ to be minimized and must be satisfied m additional conditions of the form

$$g_1(x_1, x_2, \ldots, x_n) = 0$$

$$g_2(x_1, x_2, \ldots, x_n) = 0$$

$$\vdots$$

$$g_m(x_1, x_2, \ldots, x_n) = 0$$

A new function $\overline{G}$ is formed, where

$$\overline{G} = G + \lambda_1 g_1 + \lambda_2 g_2 + \ldots + \lambda_m g_m$$

$$\overline{G} = G + \sum_{i=1}^{m} \lambda_i g_i$$

The constants λ_1, λ_2, …, λ_m are the Lagrange multipliers. When G has a stationary value, $\overline{G}$ must be such that

$$\frac{\partial \overline{G}}{\partial x_1} = 0$$

$$\frac{\partial \overline{G}}{\partial x_2} = 0$$

$$\vdots$$

$$\frac{\partial \overline{G}}{\partial x_n} = 0$$

These n conditions plus m conditions $g_1 = g_2 = \ldots = g_m = 0$ provide $m+n$ independent equations from which the $m+n$ unknowns x_1, x_2, …, x_n and λ_1, λ_2, …, λ_m can be determined.

From the previous example, if we do not introduce the equation $x = a\theta$ into the equation $U = k_1 x^2 + \dfrac{k_\alpha \alpha^2}{2}$, we have the total potential energy of the system of the form

$$V(\theta, \psi, x) = \frac{1}{2}(k_\alpha - Pa)\theta^2 - k_\alpha \theta\psi + k_1 x^2 + \frac{1}{2}(k_\alpha - Pa)\psi^2$$

with the additional condition

$$x - a\theta = 0$$

We form the new function

$$\overline{V} = V(\theta, \psi, x) + \lambda(x - a\theta)$$

where λ is the Lagrange multipliers. To be a minimum, $\overline{V}$ must satisfy the conditions

$$\frac{\partial \overline{V}}{\partial \theta} = (k_\alpha - Pa)\theta - k_\alpha \psi - a\lambda = 0$$

$$\frac{\partial \overline{V}}{\partial \psi} = -k_\alpha \theta + (k_\alpha - Pa)\psi = 0$$

$$\frac{\partial \overline{V}}{\partial x} = 2k_1 x + \lambda = 0$$

From the last condition, we find that

$$\lambda = -2k_1 x = -2k_1 a\theta$$

Substituting λ into the rest of the equations and rearrange the terms, we have

$$2c_0\theta + c_1\psi = 0$$

$$c_1\theta + 2c_2\psi = 0$$

which are obtained before.

Example 4-5: Stability analysis of beam-column

Find the critical load of the linearly elastic simply supported beam-column as shown in Fig. Ex 4-5. The beam-column is subjected to axial force P and a sinusoidal distributed load $p(x)$.

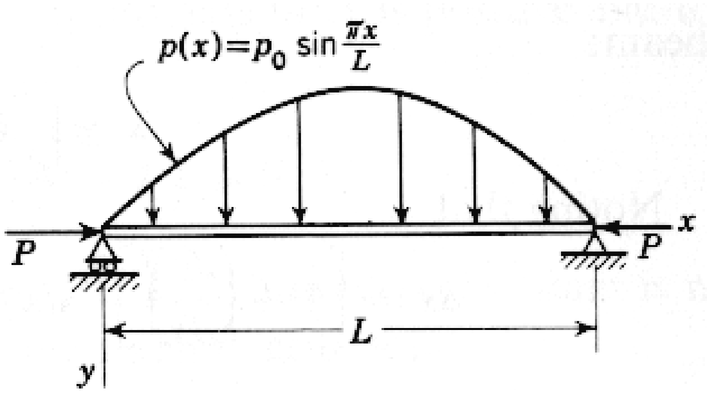


Fig. Ex 4-5

For simplicity, neglecting the shear deformation and assuming that the normal stress is given by

$$\sigma_x = \frac{N}{A} + \frac{My}{I}$$

and other stresses are zero.

From Hooke's law, $\sigma_x = E\varepsilon_x$. The strain energy in this system is

$$U = \iiint_V \frac{1}{2}\sigma_x \varepsilon_x dV$$

$$U = \iiint_V \frac{\sigma_x^2}{2E} dAdx = \frac{1}{2E}\int_0^L \left[\int_A \left(\frac{N}{A} + \frac{My}{I}\right)^2 dA\right] dx$$

Thus,

$$U = \frac{1}{2E}\int_0^L \left[\int_A \left(\frac{N^2}{A^2} + 2\frac{NM}{AI}y + \frac{M^2 y^2}{I^2}\right) dA\right] dx$$

Since $\int_A ydA = 0$ and $\int_A y^2 dA = I$,

$$U = \int_0^L \left(\frac{N^2}{2EA} + \frac{M^2}{2EI}\right) dx$$

If we are interested only the transverse displacement and the stability of the beam-column, we need to consider only the energy due to changes in curvature. Thus,

$$U = \int_0^L \frac{M^2}{2EI} dx = \frac{EI}{2}\int_0^L \left(\frac{d^2 v}{dx^2}\right)^2 dx$$

where v is the transverse displacement. Then, the total potential energy of the external forces is

$$\Omega = -\int_0^L p(x) v dx - P\Delta$$

where Δ is the displacement of P due to the changes in curvature of the beam-column.

$$\Delta \cong \int_0^L (ds - dx)$$

Since the beam-column's length $ds = \left[1 + \left(\frac{dv}{dx}\right)^2\right]^{\frac{1}{2}} dx \approx \left[1 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2\right] dx$, thus,

$$\Delta \cong \int_0^L \left(dx + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx - dx \right) = \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx$$

Hence,

$$\Omega = -\int_0^L \left[\frac{P}{2}\left(\frac{dv}{dx}\right)^2 + p(x)v \right] dx$$

The total potential energy of the beam-column is

$$V = \int_0^L \left[\frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 - \frac{P}{2}\left(\frac{dv}{dx}\right)^2 - pv \right] dx$$

Assuming that the elastic curve of the beam-column is in the for of

$$v = C \sin \frac{\pi x}{L}$$

where C is an unknown constant. This equation satisfies all the kinetic boundary conditions of the beam-column. Since the magnitude of the displacement at any point depends on the magnitude of C, thus, C is a generalized coordinate. Substituting v into the total potential energy equation, we have

$$V = \int_0^L \left[\frac{EI}{2}\left(\frac{d^2}{dx^2}\left\{ C \sin \frac{\pi x}{L} \right\}\right)^2 - \frac{P}{2}\left(\frac{d}{dx}\left\{ C \sin \frac{\pi x}{L} \right\}\right)^2 - p\left\{ C \sin \frac{\pi x}{L} \right\} \right] dx$$

$$V = \frac{L}{4}\left[\frac{C^2\pi^2}{L^2}\left(\frac{EI\pi^2}{L^2} - P \right) - 2p_o C \right]$$

If the beam-column is in equilibrium,

$$\frac{dV}{dC} = \frac{L}{4}\left[\frac{2C\pi^2}{L^2}\left(\frac{EI\pi^2}{L^2} - P \right) - 2p_o \right] = 0$$

$$C = \frac{p_o L^4}{\pi^2 (EI\pi^2 - PL^2)}$$

$$v = \frac{p_o L^4}{\pi^2 (EI\pi^2 - PL^2)} \sin\frac{\pi x}{L}$$

If the beam-column is in stable equilibrium,

$$\frac{d^2V}{dC^2} = \frac{\pi^2}{2L}\left(\frac{EI\pi^2}{L^2} - P\right) > 0$$

When P has a value that $\delta^2 V = 0$, the beam-column is no longer in stable equilibrium. Thus, the critical load of the beam-column-column is

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

which is the Euler buckling load of pinned-pinned column.

4.6 Rayleigh-Ritz Method

A continuous distributed deformable body consists of infinitely many material points. Thus, it has infinitely many degrees of freedom. The Rayleigh-Ritz method is approximation method in which the continuous systems are reduced to systems with a finite number of degrees of freedom. This method can be used to analyze deformations, stability, nonlinear behavior, and vibrations.

In Rayleigh-Ritz method, the components of displacement u, v, and w of a system are approximated by function containing a finite number of independent parameters. Then, we determine these parameters so that the total potential energy of the system computed based on the approximate displacements is minimum.

Let the components of displacement u, v, and w are of the form

$$u = a_1\phi_1(x,y,z) + a_2\phi_2(x,y,z) + \ldots + a_n\phi_n(x,y,z)$$

$$v = b_1\psi_1(x,y,z) + b_2\psi_2(x,y,z) + \ldots + b_n\psi_n(x,y,z)$$

$$w = c_1\eta_1(x,y,z) + c_2\eta_2(x,y,z) + \ldots + c_n\eta_n(x,y,z)$$

where a_1, a_2, …, a_n, b_1, b_2, …, b_n, c_1, c_2, …, c_n are $3n$ unknowns linearly independent parameters which may be called the generalized coordinates. ϕ_1, ϕ_2, …, ϕ_n, ψ_1, ψ_2, …, ψ_n, η_1, η_2, …, η_n are continuous functions of the coordinates x, y, and z which represent the modes of deformation.

The functions of the modes of deformation ϕ_1, ϕ_2, …, ϕ_n, ψ_1, ψ_2, …, ψ_n, η_1, η_2, …, η_n are chosen so that the components of displacement satisfy all of the kinematic (displacement) boundary conditions for all values of the constant parameters a_1, a_2, …, a_n, b_1, b_2, …, b_n, c_1, c_2, …, c_n, but, they do not necessarily satisfy the static (force) boundary conditions. These kinds of function are usually called *kinematically admissible functions*.

Since the components of displacement u, v, and w are defined in terms of only $3n$ independent parameters, these parameters behaves as generalized coordinates, thus, the system has only $3n$ degrees of freedom.

From the components of displacement, we can determine the approximate strains and, then, the total potential energy of the system.

$$V(u,v,w) = V(x,y,z,a_1,\ldots,a_n,b_1,\ldots,b_n,c_1,\ldots c_n)$$

The variation of the components of displacement can be written in the form of the variations in the parameters a_i, b_i, and c_i.

$$\delta u = \sum_{i=1}^{n} \phi_i \delta a_i$$

$$\delta v = \sum_{i=1}^{n} \psi_i \delta b_i$$

$$\delta w = \sum_{i=1}^{n} \eta_i \delta c_i$$

Hence, the variation of the total potential energy of the system can be written as

$$\delta V = \sum_{i=1}^{n} \left(\frac{\partial V}{\partial a_i} \delta a_i + \frac{\partial V}{\partial b_i} \delta b_i + \frac{\partial V}{\partial c_i} \delta c_i \right)$$

If the system is in equilibrium,

$$\sum_{i=1}^{n} \left(\frac{\partial V}{\partial a_i} \delta a_i + \frac{\partial V}{\partial b_i} \delta b_i + \frac{\partial V}{\partial c_i} \delta c_i \right) = 0$$

for arbitrary values of the variations δa_i, δb_i, and δc_i. Then, we have

$$\frac{\partial V}{\partial a_1} = 0 \qquad \frac{\partial V}{\partial a_2} = 0 \qquad \cdots \qquad \frac{\partial V}{\partial a_n} = 0$$

$$\frac{\partial V}{\partial b_1} = 0 \qquad \frac{\partial V}{\partial b_2} = 0 \qquad \cdots \qquad \frac{\partial V}{\partial b_n} = 0$$

$$\frac{\partial V}{\partial c_1} = 0 \qquad \frac{\partial V}{\partial c_2} = 0 \qquad \cdots \qquad \frac{\partial V}{\partial c_n} = 0$$

These equations are $3n$ linearly independent simultaneous equations which can be solved for the unknowns parameters a_i, b_i, and c_i.After solving for the unknowns parameters a_i, b_i, and c_i, we then obtain the approximate displacement functions.

In the case of stability analysis, the equations are homogeneous, we then determine the buckling loads by setting the determinant of the coefficients to be zero.

Some important characteristics of the Rayleigh-Ritz method

1. The accuracy of the assumed displacement is in general increased with an increase in the number of parameters. However, the exact solutions are rarely obtained.
2. Since the differential equations of equilibrium do not enter the analysis, the equilibrium is satisfied in an average sense through minimization of the total potential energy. Thus, in general, the stresses computed do not satisfy the equations of equilibrium.
3. Although the Rayleigh-Ritz method may provide fairly accurate results for the displacements, the corresponding stresses may differ significantly from their exact values since the stresses depend on the derivatives of the displacements.
4. Since the Rayleigh-Ritz method use a finite number of degree of freedom to approximate the displacements of the system having infinitely many degree of

freedom, the approximate system is less flexible than the actual system. In the stability analysis, the Rayleigh-Ritz method always produce the buckling loads that larger than the exact values.

5. The modes of deformation are often taken as polynomial or trigonometric functions since they are easy to manipulate.
6. The approximate displacement functions should not omit terms of lower order since it prevents the convergence of the solutions.

Example 4-6

Find the maximum deflection and moment of the simply supported beam as shown in Fig. Ex 4-6 by using the Rayleigh-Ritz method and compared with the results from mechanic of material method of analysis.

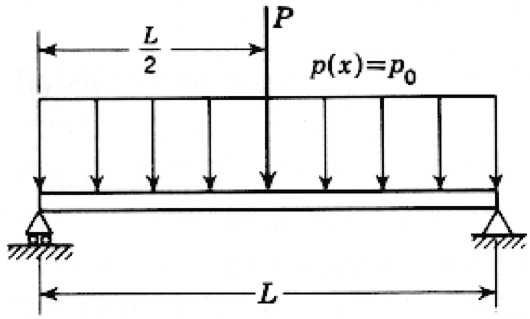


Fig. Ex 4-6

Consider the simply supported beam as shown in Fig. Ex 4-6. The elastic curve of the beam can be assumed as a sine function of

$$v = a\sin\frac{\pi x}{L}$$

where a is an unknown constants. Note that $\sin\frac{\pi x}{L}$ satisfy the boundary conditions at $x = 0$ and $x = L$.

If only the strain energy due to bending is considered, we have the total potential energy of the beam is of the form

$$V = \frac{EI}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx - \int_0^L p(x)v dx - Pv\Big|_{L/2}$$

$$V = \frac{EI\pi^4}{4L^3}a^2 - 2p_o\frac{L}{\pi}a - Pa$$

Now, we choose the constant a so as to minimize V.

$$\frac{\partial V}{\partial a} = \frac{EI\pi^4}{2L^3}a - 2p_o\frac{L}{\pi} - P = 0$$

$$a = \frac{4p_oL^4}{\pi^5 EI} + \frac{2PL^3}{\pi^4 EI} = \frac{2L^3(2p_o + \pi P)}{\pi^5 EI}$$

$$v = \frac{2L^3(2p_o + \pi P)}{\pi^5 EI}\sin\frac{\pi x}{L}$$

Evaluating the deflection v of the beam at $x = L/2$, we have

$$v\Big|_{L/2} = a = \frac{4p_oL^4}{\pi^5 EI} + \frac{2PL^3}{\pi^4 EI} = \frac{p_oL^4}{76.5EI} + \frac{PL^3}{48.7EI}$$

By using the mechanics of materials, we obtain

$$v\Big|_{L/2} = \frac{p_o L^4}{76.8EI} + \frac{PL^3}{48EI}$$

Thus, by using only one parameter, we obtain a maximum deflection, which is only 0.39% in error in case of uniformly distributed load and 1.46% in error in case of concentrated load. However, the approximate deflection curve gives a bending moment at $x = L/2$ of

$$M\Big|_{L/2} = -EI\frac{d^2v}{dx^2} = \frac{4p_o L^2}{\pi^3} + \frac{2PL}{\pi^2} = \frac{p_o L^2}{7.75} + \frac{PL}{4.93}$$

The first term has 3.15% in error and the second term has 23.37% in error. Note that the normal stresses are proportion to the bending moment, thus, are in error by the same percentages.

To obtain more accurate results, let us use two-parameter approximation elastic curve of the beam as

$$v = a\sin\frac{\pi x}{L} + b\sin\frac{3\pi x}{L}$$

Note that this function also satisfy the kinematic boundary conditions of the beam. Performing the determination of the total potential energy of the beam as before, and then from the conditions $\frac{\partial V}{\partial a} = 0$ and $\frac{\partial V}{\partial b} = 0$, we find that

$$a = \frac{4p_o L^4}{\pi^5 EI} + \frac{2PL^3}{\pi^4 EI}$$

$$b = \frac{4p_o L^4}{243\pi^5 EI} - \frac{2PL^3}{81\pi^4 EI}$$

In this case, the maximum deflection at $x = L/2$ is

$$v\Big|_{L/2} = \frac{p_o L^4}{76.8EI} + \frac{PL^3}{48.1EI}$$

which coincide with the exact solution. The bending moment at $x = L/2$ is

$$M\Big|_{L/2} = \frac{p_o L^2}{8.05} + \frac{PL}{4.44}$$

The first term has now only 0.63% in error and the second term has 11.00% in error. We can further reduce the error by increasing the number of parameters in the approximate displacement function.

Example 4-7

Determine the buckling loads of the fixed end column as shown in Fig. Ex 4-7 by using the Rayleigh-Ritz method.

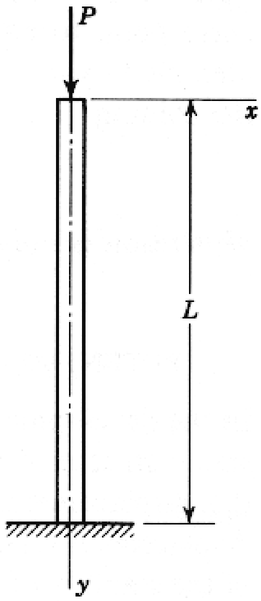


Fig. Ex 4-7

Consider the fixed end column as shown in Fig. Ex 4-7. The transverse deflection of the column can be assumed to be of the form

$$v = a(x^3 - 3xL^2 + 2L^3) + b(x-L)^2$$

where a and b are the unknown parameters. This function satisfy the boundary condition of the column at $x = L$. If we consider only the strain energy of the bending moment, we have

$$U = \frac{EI}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx$$

The potential energy of the external force P is

$$\Omega = -\frac{P}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx$$

Thus, the total potential energy of the column is

$$V = \frac{EI}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx - \frac{P}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx$$

$$V = 6a^2L^3(EI - \frac{2}{5}PL^2) + abL^2(6EI - \frac{5}{2}PL^2) + 2b^2L(EI - \frac{1}{3}PL^2)$$

If the parameter a and b are to correspond to a stationary value of V,

$$\frac{\partial V}{\partial a} = 12L^3(EI - \frac{2}{5}PL^2)a + L^2(6EI - \frac{5}{2}PL^2)b = 0$$

$$\frac{\partial V}{\partial b} = L^2(6EI - \frac{5}{2}PL^2)a + 4L(EI - \frac{1}{3}PL^2)b = 0$$

A nontrivial solution to this column of homogeneous equations exists only if the determinant of the coefficients vanishes. Then, we have

$$3P^2 - 104\frac{EI}{L^2}P + 240\frac{(EI)^2}{L^4} = 0$$

Solving the polynomial equation, we obtain

$$P_1 = 2.486\frac{EI}{L^2}$$

$$P_2 = 32.181\frac{EI}{L^2}$$

Thus, the buckling load of the column is $P_{cr} = 2.486\frac{EI}{L^2}$ which is only 0.75% larger than the exact value of $\frac{\pi^2 EI}{4L^2}$.

4.7 Introduction to Finite Element Method

In the Rayleigh-Ritz method, each of the modes of deformation ϕ_1, ϕ_2, …, ϕ_n, ψ_1, ψ_2, …, ψ_n, η_1, η_2, …, η_n span the entire structure and the generalized coordinates a_1, a_2, …, a_n, b_1, b_2, …, b_n, c_1, c_2, …, c_n are usually have no physical meaning.

In the finite element method, there are many approximate functions, each comparatively simple and each spanning a limited region of the structure. In addition the degree of freedom are the actual displacements of specific points instead of generalized coordinates.

Beam Element Formulation

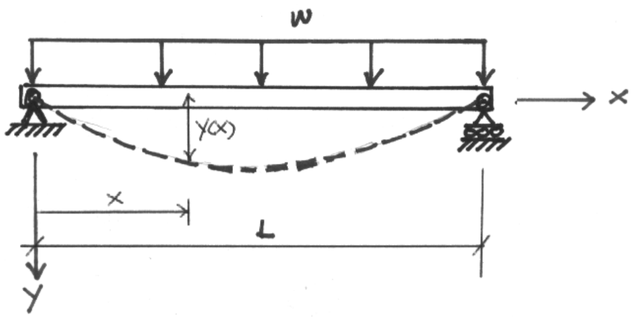


Fig. 4.7

Consider the simply supported prismatic beam having a constant stiffness EI as shown in Fig. 4.7. First, consider the whole beam as an element in order to see the error that is occurred due to a coarse element. Then, we will divide the beam into more segments and redo the analysis.

Selecting an approximate displacement function that closes to the actual displacement function of the beam in a form of polynomial function

$$\bar{y}(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

This function must satisfy the geometrical (displacement) boundary conditions of the beam.

At $x = 0$, $\bar{y}(0) = 0$, $\qquad \alpha_1 = 0$

At $x = L$, $\bar{y}(L) = \alpha_2(L) + \alpha_3(L^2) + \alpha_4(L^3) = 0$,

$$\alpha_2 = -\alpha_3 L - \alpha_4 L^2$$

Thus,

$$\bar{y}(x) = \alpha_3(x^2 - Lx) + \alpha_4(x^3 - L^2 x)$$

The external potential energy of the beam is

$$\Omega = -\int_0^L w_o y \; dx = \left[\alpha_3 \frac{L^3}{6} + \alpha_4 \frac{L^4}{4}\right] w_o$$

The strain energy of the beam is

$$U = \frac{1}{2}EI\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx$$

From the assumed shape function,

$$\bar{y}(x) = \alpha_3(x^2 - Lx) + \alpha_4(x^3 - L^2x)$$

$$\frac{d\bar{y}}{dx} = \alpha_3(2x - L) + \alpha_4(3x^2 - L^2)$$

$$\frac{d^2\bar{y}}{dx^2} = 2\alpha_3 + 6x\alpha_4$$

Thus, $$U = EI(2\alpha_3^2 L + 6\alpha_3\alpha_4 L^2 + 6\alpha_4^2 L^3)$$

The total potential energy of the beam is

$$V = w_o\left(\alpha_3\frac{L^3}{6} + \alpha_4\frac{L^4}{4}\right) + EI(2\alpha_3^2 L + 6\alpha_3\alpha_4 L^2 + 6\alpha_4^2 L^3)$$

For the static equilibrium, the total potential energy of the beam must be minimum.

$\frac{\partial V}{\partial \alpha_3} = 0$; $$2EI\alpha_3 + 3EIL\alpha_4 = -\frac{w_o L^2}{12}$$

$\frac{\partial V}{\partial \alpha_4} = 0$; $$2EI\alpha_3 + 4EIL\alpha_4 = -\frac{w_o L^2}{12}$$

Solving the simultaneous equations for α_3 and α_4, we have $\alpha_3 = -\frac{w_o L^2}{24EI}$ and $\alpha_4 = 0$

Since $\alpha_2 = -\alpha_3 L - \alpha_4 L^2$, thus

$$\alpha_2 = \frac{w_o L^3}{24EI}$$

Substituting α_1 to α_4 into the assumed shape function, we have

$$\bar{y}(x) = \frac{w_o}{24EI}(-L^2x^2 + L^3x)$$

Comparing the result with the solution by the mechanics of materials,

$$y(x) = \frac{w_o}{24EI}(x^4 - 2L^2x^3 - L^3x)$$

At the mid-span of the beam, we have the deflection by the approximated method and the deflection by the classical method equal to

$$\bar{y}\left(\frac{L}{2}\right) = \frac{1}{96}\frac{w_o L^4}{EI}$$

$$y(\frac{L}{2}) = \frac{5}{384}\frac{w_o L^4}{EI}$$

The error between these two methods is about 20%. However, we can improve the solution by using a sine function as previously shown. It should be noted that if we use a higher degree of the polynomial function, the solution is the same due to the nature of the beam structure.

Compatibility

Another way to increase the accuracy of the solution is to break the beam down into more pieces as an example shown in Fig. 4.8.

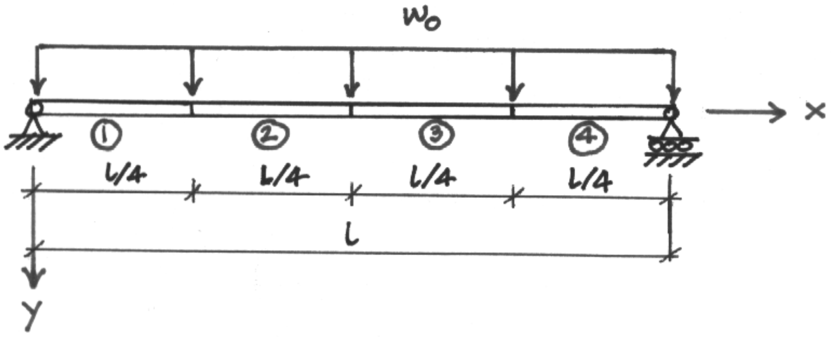


Fig. 4.8

If we choose to work this way, it is conceivable that each element will have a solution function that is different from the others. If we are to assemble these elements in a mathematical sense, there must be some compatibility requirements placed on the function of the adjacent elements. This function is usually called the compatibility condition of the beam elements.

The compatibility conditions of a structure require that

1. Within each region, the displacement varies smoothly with no discontinuity.
2. At the boundary between neighboring regions, the displacement matches each other in a manner consistent with the problem under consideration.
3. At the boundary of the whole structure, the prescribed displacement boundary conditions such as support conditions are satisfied.

The shape functions of the element i and $i+1$ of the beam in Fig. 4.9 are

$$y^i = \alpha_1^i + \alpha_2^i x + \alpha_3^i x^2 + \alpha_4^i x^3$$

$$y^{i+1} = \alpha_1^{i+1} + \alpha_2^{i+1} x + \alpha_3^{i+1} x^2 + \alpha_4^{i+1} x^3$$

Thus, the compatibility conditions of the junction between the element i and $i+1$ of the beam are

$$y^i(\frac{l}{n}) = y^{i+1}(0)$$

$$\left.\frac{dy^i}{dx}\right|_{\frac{l}{n}} = \left.\frac{dy^{i+1}}{dx}\right|_0$$

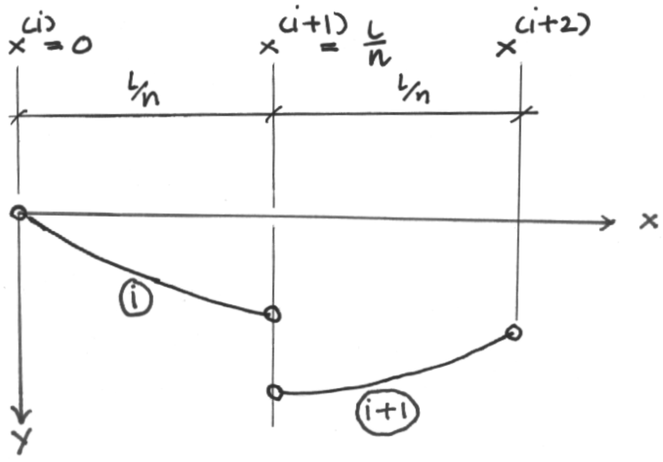


Fig. 4.9

In general case, let consider the element i of a beam having the degree of freedom q_1 to q_4 as shown in Fig. 4.10. The shape function of the element i of the beam can be expressed as

$$y = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3$$

$$\frac{dy}{d\xi} = \alpha_2 + 2\alpha_3\xi + 3\alpha_4\xi^2$$

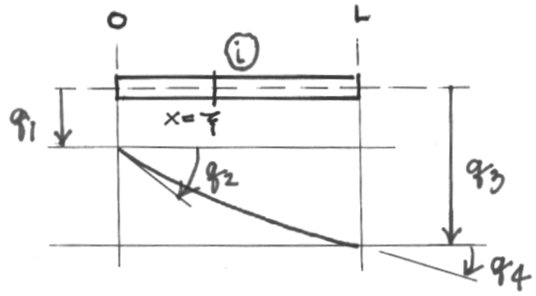


Fig. 4.10

At $\xi = 0$, $y = q_1$ and $\frac{dy}{d\xi} = q_2$. Thus,

$$q_1 = \alpha_1 + \alpha_2(0) + \alpha_3(0) + \alpha_4(0)$$

$$q_2 = \alpha_1(0) + \alpha_2 + \alpha_3(0) + \alpha_4(0)$$

At $\xi = L$, $y = q_3$ and $\frac{dy}{d\xi} = q_4$. Thus,

$$q_3 = \alpha_1 + \alpha_2 L + \alpha_3 L^2 + \alpha_4 L^3$$

$$q_4 = \alpha_1(0) + \alpha_2 + 2\alpha_3 L + 3\alpha_4 L^2$$

In the matrix form,

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

This matrix expressing the relationship between the displacement q of the element i and the generalized coordinate α of the shape function. It has a physical interpretation as following:

1. When $\alpha_1 = 1$ and other $\alpha = 0$, $q_1 = 1$, $q_2 = 0$, $q_3 = 1$, and $q_4 = 0$
2. When $\alpha_2 = 1$ and other $\alpha = 0$, $q_1 = 0$, $q_2 = 1$, $q_3 = L$, and $q_4 = 1$
3. When $\alpha_3 = 1$ and other $\alpha = 0$, $q_1 = 0$, $q_2 = 0$, $q_3 = L^2$, and $q_4 = 2L$
4. When $\alpha_4 = 1$ and other $\alpha = 0$, $q_1 = 0$, $q_2 = 0$, $q_3 = 2L$, and $q_4 = 3L^2$

These interpretations can be presented graphically as shown in Fig. 4.11.

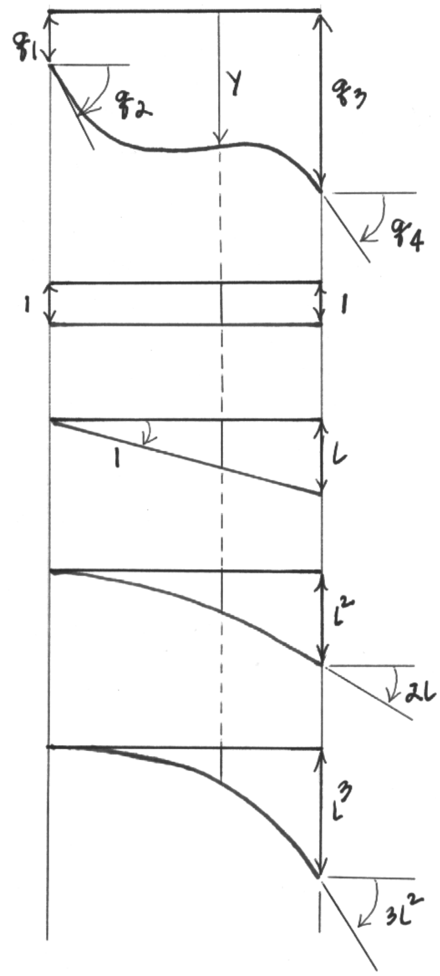


Fig. 4.11

Shape Function of the Beam Element

If we inverse the relationship between the displacement q of the element i and the generalized coordinate α of the shape function, we have

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \frac{1}{L^3}\begin{bmatrix} L^3 & 0 & 0 & 0 \\ 0 & L^3 & 0 & 0 \\ -3L & -2L^2 & 3L & -L^2 \\ 2 & L & -2 & L \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

or symbolically,

$$\{\alpha\} = [T]\{q\}$$

The shape function of the beam element can be written in the matrix form as

$$y = \lfloor 1 \quad x \quad x^2 \quad x^3 \rfloor \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

Substitute $\{\alpha\} = [T]\{q\}$ into the shape function, we have

$$y = \lfloor 1 \quad x \quad x^2 \quad x^3 \rfloor \frac{1}{L^3}\begin{bmatrix} L^3 & 0 & 0 & 0 \\ 0 & L^3 & 0 & 0 \\ -3L & -2L^2 & 3L & -L^2 \\ 2 & L & -2 & L \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$y = \lfloor N_1 \quad N_2 \quad N_3 \quad N_4 \rfloor \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

or
$$y = \lfloor N \rfloor \{q\} = \{q\}^T \{N\}$$

where N_i is called the *shape functions* or *interpolation functions* and

$$N_1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

$$N_2 = x - 2\left(\frac{x^2}{L}\right) + \left(\frac{x^3}{L^2}\right)$$

$$N_3 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$N_4 = -\left(\frac{x^2}{L}\right) + \left(\frac{x^3}{L^2}\right)$$

If we plot the shape function N_i with respect to the coordinate x, we obtain 4 curves as shown in Fig. 4.12. Physically, each of the 4 shape functions represents the deflection curve for the beam element produced by setting the corresponding degree of freedom to be one and the others to be zero.

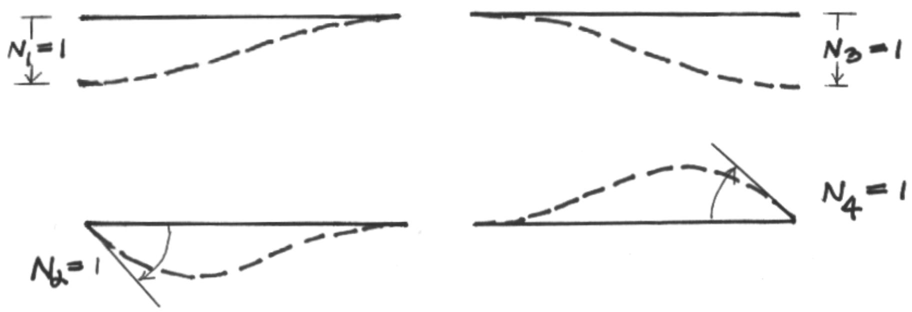


Fig. 4.12

The first differentiation (slope) of the shape function of the beam element can be obtained as

$$\frac{dy}{dx} = \left\lfloor \frac{dN}{dx} \right\rfloor \{q\}$$

where

$$\left\lfloor \frac{dN}{dx} \right\rfloor = \left\lfloor -\frac{6x}{L^2} + \frac{6x^2}{L^3} \quad 1 - \frac{4x}{L} + \frac{3x^2}{L^2} \quad \frac{6x}{L^2} - \frac{6x^2}{L^3} \quad -\frac{2x}{L} + \frac{3x^2}{L^2} \right\rfloor$$

The second differentiation (curvature) of the shape function of the beam element can be obtained as

$$\frac{d^2y}{dx^2} = \left\lfloor \frac{d^2N}{dx^2} \right\rfloor \{q\} = \{q\}^T \left\{ \frac{d^2N}{dx^2} \right\}$$

where

$$\left\lfloor \frac{d^2N}{dx^2} \right\rfloor = \left\lfloor -\frac{6}{L^2} + \frac{12x}{L^3} \quad -\frac{4}{L} + \frac{6x}{L^2} \quad \frac{6}{L^2} - \frac{12x}{L^3} \quad -\frac{2}{L} + \frac{6x}{L^2} \right\rfloor$$

Let us consider the simply supported beam subjected to a uniformly distributed load as shown in Fig. 4.13.

By using the finite element method, we divide the beam into four segments. The positive degrees of freedom x_i of each node of the beam are specified as shown. It should be noted that in this case we use the notation of the degree of freedom x_i replacing q_i since q_i is the local degree of freedom of the beam element. However, when we consider the whole beam, we need to use the global degree of freedom x_i of the beam.

The total potential energy of a beam element i can be determined from the equation

$$V^{(i)} = U^{(i)} + \Omega^{(i)}$$

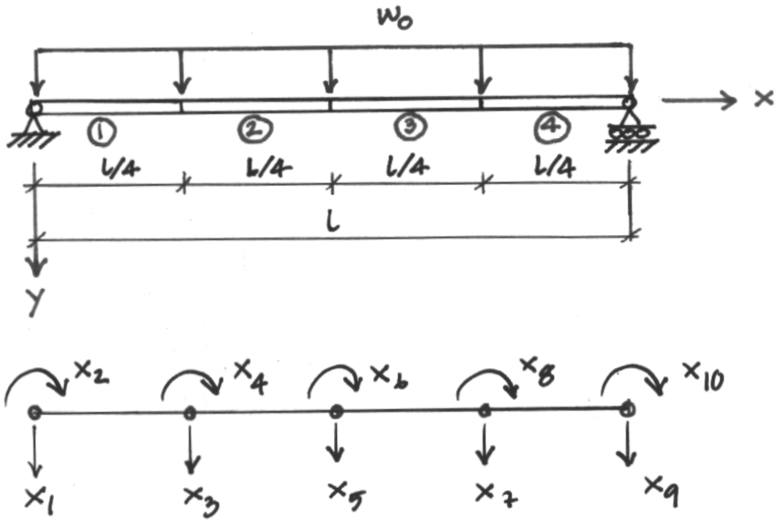


Fig. 4.13

Stiffness Equation of the Beam Element

The strain energy of the beam element can be written in the form of

$$U = \frac{1}{2} EI \int_0^L \left(\frac{d^2 y}{dx^2} \right)^2 dx$$

Rearranging the equation,

$$\begin{aligned}
U &= \frac{1}{2} \int_0^L \left(\frac{d^2 y}{dx^2} \right) EI \left(\frac{d^2 y}{dx^2} \right) dx \\
&= \frac{1}{2} \int_0^L \{q\}^T \left\{ \frac{d^2 N}{dx^2} \right\} EI \left\lfloor \frac{d^2 N}{dx^2} \right\rfloor \{q\} dx \\
&= \frac{1}{2} \{q\}^T \left(\int_0^L \left\{ \frac{d^2 N}{dx^2} \right\} EI \left\lfloor \frac{d^2 N}{dx^2} \right\rfloor dx \right) \{q\} \\
&= \frac{1}{2} \{q\}^T [k] \{q\}
\end{aligned}$$

where

$$[k] = \frac{EI}{L} \begin{bmatrix} \frac{12}{L^2} & \frac{6}{L} & -\frac{12}{L^2} & \frac{6}{L} \\ \frac{6}{L} & 4 & -\frac{6}{L} & 2 \\ -\frac{12}{L^2} & -\frac{6}{L} & \frac{12}{L^2} & -\frac{6}{L} \\ \frac{6}{L} & 2 & -\frac{6}{L} & 4 \end{bmatrix}$$

The $[k]$ matrix is the called the stiffness matrix of the beam element.

The potential energy due to external load of the beam element can be determined from the equation

$$\Omega = -\int_0^L w_o y dy$$

where $y = \lfloor N \rfloor \{q\} = \{q\}^T \{N\}$. Therefore,

$$\Omega = -\int_0^L \{q\}^T w_o \{N\} dx = -\{q\}^T \int_0^L w_o \{N\} dx$$

Let $\int_0^L w_o \{N\} dx = \{Q\}$, then

$$\Omega = -\{q\}^T \{Q\}$$

For the beam in this case, we have

$$\{Q\} = w_o \int_0^{L/4} \{N\} dx = w_o \left\{ \begin{matrix} x - \dfrac{x^3}{L^2} + \dfrac{1}{2}\dfrac{x^4}{L^3} \\ \dfrac{1}{2}x^2 - \dfrac{2}{3}\dfrac{x^3}{L} + \dfrac{1}{4}\dfrac{x^4}{L^2} \\ \dfrac{x^3}{L^2} - \dfrac{1}{2}\dfrac{x^4}{L^3} \\ -\dfrac{1}{3}\dfrac{x^3}{L} + \dfrac{1}{4}\dfrac{x^4}{L^2} \end{matrix} \right\}_0^{L/4}$$

Therefore, the total potential energy of a beam element i can be determined from

$$V^{(i)} = \frac{1}{2}\{q\}^T [k] \{q\} - \{q\}^T \{Q\}$$

From the principle of stationary (minimum) potential energy, we can determine the value of the degree of freedom of each node of the beam.

$$\sum \frac{\partial V^{(i)}}{\partial q_i} = \frac{\partial}{\partial q_i} \sum V^{(i)} = 0$$

Let us consider the element 2 of the beam, the strain energy stored in this element is

$$U^{(2)} = \frac{1}{2} \begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix}^T \begin{bmatrix} k_{11}^2 & k_{12}^2 & k_{13}^2 & k_{14}^2 \\ k_{21}^2 & k_{22}^2 & k_{23}^2 & k_{24}^2 \\ k_{31}^2 & k_{32}^2 & k_{33}^2 & k_{34}^2 \\ k_{41}^2 & k_{42}^2 & k_{43}^2 & k_{44}^2 \end{bmatrix} \begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix}$$

Then, the partial derivative of the strain energy with respect to the degree of freedom x_3 to x_6 can be determined as following

$$\frac{\partial U^{(2)}}{\partial x_3} = \frac{1}{2}\begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} k_{11}^2 & k_{12}^2 & k_{13}^2 & k_{14}^2 \\ k_{21}^2 & k_{22}^2 & k_{23}^2 & k_{24}^2 \\ k_{31}^2 & k_{32}^2 & k_{33}^2 & k_{34}^2 \\ k_{41}^2 & k_{42}^2 & k_{43}^2 & k_{44}^2 \end{bmatrix} \begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} +$$

$$\frac{1}{2}\begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} +$$

$$\frac{1}{2}\begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix}^T \begin{bmatrix} k_{11}^2 & k_{12}^2 & k_{13}^2 & k_{14}^2 \\ k_{21}^2 & k_{22}^2 & k_{23}^2 & k_{24}^2 \\ k_{31}^2 & k_{32}^2 & k_{33}^2 & k_{34}^2 \\ k_{41}^2 & k_{42}^2 & k_{43}^2 & k_{44}^2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\frac{\partial U^{(2)}}{\partial x_3} = \begin{Bmatrix} k_{11}^2 \\ k_{12}^2 \\ k_{13}^2 \\ k_{14}^2 \end{Bmatrix}^T \begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix}$$

$$\frac{\partial U^{(2)}}{\partial x_4} = \begin{Bmatrix} k_{21}^2 \\ k_{22}^2 \\ k_{23}^2 \\ k_{24}^2 \end{Bmatrix}^T \begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} k_{12}^2 \\ k_{22}^2 \\ k_{32}^2 \\ k_{42}^2 \end{Bmatrix}^T \begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix}$$

$$\frac{\partial U^{(2)}}{\partial x_5} = \begin{Bmatrix} k_{31}^2 \\ k_{32}^2 \\ k_{33}^2 \\ k_{34}^2 \end{Bmatrix}^T \begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} k_{13}^2 \\ k_{23}^2 \\ k_{33}^2 \\ k_{43}^2 \end{Bmatrix}^T \begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix}$$

$$\frac{\partial U^{(2)}}{\partial x_6} = \begin{Bmatrix} k_{14}^2 \\ k_{24}^2 \\ k_{34}^2 \\ k_{44}^2 \end{Bmatrix}^T \begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} k_{41}^2 \\ k_{42}^2 \\ k_{43}^2 \\ k_{44}^2 \end{Bmatrix}^T \begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix}$$

The external potential energy of the element 2 is

$$\Omega^{(2)} = -\begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix}^T \begin{Bmatrix} Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix}$$

Then, the partial derivative of the external potential energy with respect to the degree of freedom x_3 to x_6 can be determined as following

$$\frac{\partial \Omega^{(2)}}{\partial x_3} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}^T \begin{Bmatrix} Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} + 0 = -Q_3$$

$$\frac{\partial \Omega^{(2)}}{\partial x_4} = -Q_4$$

$$\frac{\partial \Omega^{(2)}}{\partial x_5} = -Q_5$$

$$\frac{\partial \Omega^{(2)}}{\partial x_6} = -Q_6$$

Thus, the partial derivative of the total potential energy with respect to the degree of freedom x_3 to x_6 can be found as

$$\begin{Bmatrix} \dfrac{\partial V^{(2)}}{\partial x_3} \\ \dfrac{\partial V^{(2)}}{\partial x_4} \\ \dfrac{\partial V^{(2)}}{\partial x_5} \\ \dfrac{\partial V^{(2)}}{\partial x_6} \end{Bmatrix} = \begin{bmatrix} k_{11}^2 & k_{12}^2 & k_{13}^2 & k_{14}^2 \\ k_{21}^2 & k_{22}^2 & k_{23}^2 & k_{24}^2 \\ k_{31}^2 & k_{32}^2 & k_{33}^2 & k_{34}^2 \\ k_{41}^2 & k_{42}^2 & k_{43}^3 & k_{44}^2 \end{bmatrix} \begin{Bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} - \begin{Bmatrix} Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix}$$

Rewriting the strain energy and the external potential energy of the element 2 in the global coordinate, we have

$$\begin{Bmatrix} U^{(2)}, x_1 \\ U^{(2)}, x_2 \\ U^{(2)}, x_3 \\ U^{(2)}, x_4 \\ U^{(2)}, x_5 \\ U^{(2)}, x_6 \\ U^{(2)}, x_7 \\ U^{(2)}, x_8 \\ U^{(2)}, x_9 \\ U^{(2)}, x_{10} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11}^2 & k_{12}^2 & k_{13}^2 & k_{14}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{21}^2 & k_{22}^2 & k_{23}^2 & k_{24}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 & k_{34}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{41}^2 & k_{42}^2 & k_{43}^2 & k_{44}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{Bmatrix}$$

and

$$\left\{\Omega^{(2)},x_i\right\} = \begin{Bmatrix} \Omega^{(2)},x_1 \\ \Omega^{(2)},x_2 \\ \Omega^{(2)},x_3 \\ \Omega^{(2)},x_4 \\ \Omega^{(2)},x_5 \\ \Omega^{(2)},x_6 \\ \Omega^{(2)},x_7 \\ \Omega^{(2)},x_8 \\ \Omega^{(2)},x_9 \\ \Omega^{(2)},x_{10} \end{Bmatrix} = -\begin{Bmatrix} 0 \\ 0 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

In general, we have

$$\left\{U^{(i)},x_i\right\}_{10\mathbf{x}1} = [k^i]_{10\mathbf{x}10}\left\{x\right\}_{10\mathbf{x}1}$$

$$\left\{\Omega^{(i)},x_i\right\}_{10\mathbf{x}1} = \left\{Q\right\}_{10\mathbf{x}1}$$

and the total potential energy of all four elements of the beam is

$$\left\{V\right\} = \sum_{i=1}^{4}[U^{(i)} + \Omega^{(i)}]$$

For equilibrium, the partial derivative of the total potential energy of the beam with respect to the degree of freedom x_i must be zero.

$$\left\{V,x_i\right\} = \sum_{i=1}^{4}[U^{(i)},x_i + \Omega^{(i)},x_i] = \left\{0\right\}$$

$$\left\{V,x_i\right\} = \sum_{i=1}^{4}[k^i]\{x\} + \sum_{i=1}^{4}\left\{Q\right\} = \left\{0\right\}$$

where $\sum_{i=1}^{4}[k^i]$ is the global stiffness matrix of the beam and $\sum_{i=1}^{4}\{Q\}$ is the load vector of the beam. Symbolically, the global stiffness matrix $\sum_{i=1}^{4}[k^i]$ is usually written by using $[K]$. In details, the partial derivative of the total potential energy of the beam with respect to the degree of freedom x_i can be written as shown in the next page. In this equation, since the supports are pin and roller, the degrees of freedom x_1 and x_9 are known to be zero. It should be noted that the obtained global stiffness matrix $[K]$ is the same as one that we found by using the matrix structural analysis.

$$
\begin{Bmatrix} V,x_1 \\ V,x_2 \\ V,x_3 \\ V,x_4 \\ V,x_5 \\ V,x_6 \\ V,x_7 \\ V,x_8 \\ V,x_9 \\ V,x_{10} \end{Bmatrix}
=
\begin{bmatrix}
k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 & k_{34}^1 + k_{12}^2 & k_{13}^2 & k_{14}^2 & 0 & 0 & 0 & 0 \\
k_{41}^1 & k_{42}^1 & k_{43}^1 + k_{21}^2 & k_{44}^1 + k_{22}^2 & k_{23}^2 & k_{24}^2 & 0 & 0 & 0 & 0 \\
0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 + k_{11}^3 & k_{34}^2 + k_{12}^3 & k_{13}^3 & k_{14}^3 & 0 & 0 \\
0 & 0 & k_{41}^2 & k_{42}^2 & k_{43}^2 + k_{21}^3 & k_{44}^2 + k_{22}^3 & k_{23}^3 & k_{24}^3 & 0 & 0 \\
0 & 0 & 0 & 0 & k_{31}^3 & k_{32}^3 & k_{33}^3 + k_{11}^4 & k_{34}^3 + k_{12}^4 & k_{13}^4 & k_{14}^4 \\
0 & 0 & 0 & 0 & k_{41}^3 & k_{42}^3 & k_{43}^3 + k_{21}^4 & k_{44}^3 + k_{22}^4 & k_{23}^4 & k_{24}^4 \\
0 & 0 & 0 & 0 & 0 & 0 & k_{31}^4 & k_{32}^4 & k_{33}^4 & k_{34}^4 \\
0 & 0 & 0 & 0 & 0 & 0 & k_{41}^4 & k_{42}^4 & k_{43}^4 & k_{44}^4
\end{bmatrix}
\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{Bmatrix}
-
\begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ Q_3^{(1)} + Q_1^{(2)} \\ Q_4^{(1)} + Q_2^{(2)} \\ Q_3^{(2)} + Q_1^{(3)} \\ Q_4^{(2)} + Q_2^{(3)} \\ Q_3^{(3)} + Q_1^{(4)} \\ Q_4^{(3)} + Q_2^{(4)} \\ Q_3^{(4)} \\ Q_4^{(4)} \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

Let us redefine the degree of freedom of the beam from x_i to x_i' as shown in the figure. The degree of freedom x_i' is arranged so that the known degree of freedom (displacement at the supports) are numbered first and followed by the unknown degree of freedom. This kind of set-up will help us to partition the global stiffness matrix, which will ease the matrix manipulation.

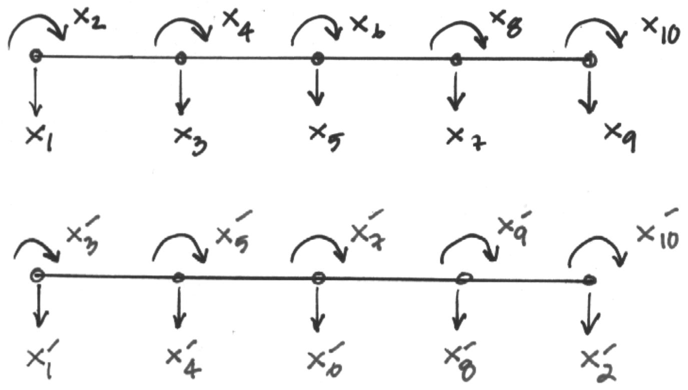


Fig. 4.14

From the Fig. 4.14, we can relate the degree of freedom x_i with the degree of freedom x_i' by the matrix

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \\ x_6' \\ x_7' \\ x_8' \\ x_9' \\ x_{10}' \end{Bmatrix}$$

In the symbolic form,

$$\{x\} = [T]\{x'\}$$

We usually call matrix $[T]$ as transformation matrix. Then, we can rewrite the total potential energy of the beam based on the new global degree of freedom x_i' as

$$\begin{aligned} V &= \frac{1}{2}\{x\}^T[K]\{x\} - \{x\}^T\{Q\} \\ &= \frac{1}{2}\{x'\}^T[T]^T[K][T]\{x'\} - \{x'\}^T[T]\{Q\} \\ &= \frac{1}{2}\{x'\}^T[K']\{x'\} - \{x'\}^T\{Q'\} \end{aligned}$$

where $\left[K'\right]=\left[T\right]^{T}\left[K\right]\left[T\right]$ and $\left\{Q'\right\}=\left[T\right]\left\{Q\right\}$

Taking the partial derivative of the total potential energy of the beam with respect to the degree of freedom x_i' and setting the result to be zero, we have

$$\left\{\frac{\partial V}{\partial x_i'}\right\}=\left\{V,x_i'\right\}=\left[K'\right]\left\{x'\right\}-\left\{P'\right\}=0$$

Partitioning the global stiffness matrix by separating the known displacements at the supports from the unknown displacements, we have

$$\begin{bmatrix} K'_{R\Delta} & K'_{Rx} \\ K'_{P\Delta} & K'_{Px} \end{bmatrix}\begin{Bmatrix} \Delta \\ x' \end{Bmatrix}-\begin{Bmatrix} P'_s \\ P' \end{Bmatrix}+\begin{Bmatrix} R \\ 0 \end{Bmatrix}=\left\{0\right\}$$

where Δ is the known support settlements, x' is the unknown displacements, P'_s is the known external loads acting on the supports, P' is the known external loads, and R is the unknown support reactions. It should be noted that $\left[K'\right]$ is a symmetric matrix and $\left[K'_{P\Delta}\right]=\left[K'_{Rx}\right]^{T}$ and vice versa. Thus, we can find the unknown displacements x' from the equation

$$\left[K'_{P\Delta}\right]\left\{\Delta\right\}+\left[K'_{Px}\right]\left\{x'\right\}-\left\{P'\right\}=\left\{0\right\}$$

$$\left\{x'\right\}=\left[K'_{Px}\right]^{-1}\left\{\left\{P'\right\}-\left[K'_{P\Delta}\right]\left\{\Delta\right\}\right\}$$

Then, we can solve for the unknown support reactions from

$$\left[K'_{R\Delta}\right]\left\{\Delta\right\}+\left[K'_{Rx}\right]\left\{x'\right\}-\left\{P'_s\right\}+\left\{R\right\}=\left\{0\right\}$$

$$\left\{R\right\}=-\left[K'_{R\Delta}\right]\left\{\Delta\right\}-\left[K'_{Rx}\right]\left\{x'\right\}+\left\{P'_s\right\}$$

Chapter 5

Static Failure and Failure Criteria

5.1 Definition of Failure

Failure can be defined as *any changes in the size, shape or material properties of a structure or mechanical part the render it incapable of satisfactorily performing its intended functions*.

Failure can be caused by the following agents:

Forces: steady, dynamic, transient, cyclic, random

Time: very short, short, long time

Temperature: low, elevated, room, steady, random, cyclic, transient

Environment: Chemical, nuclear, rain, sand

5.2 Modes of Failure

When a structural member is subjected to loads, its response depends not only on the type of material, but also on the types of loads and environment conditions. Thus, the modes of failure can be classified as

Yielding failure – The plastic deformation in the structure under operational loads that is large enough to interfere with the ability of the structure to satisfactorily performing its intended functions as an example shown in Fig. 5.1.

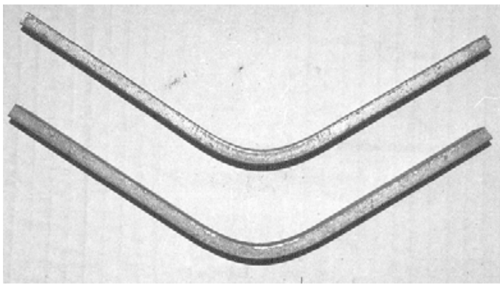


Fig. 5.1

Force induced elastic deformation – The elastic deformation that is recoverable in a structure under operational loads become large enough to interfere with the ability of the structure to satisfactorily performing its intended functions such as stiffness loss as an example shown in Fig. 5.2.

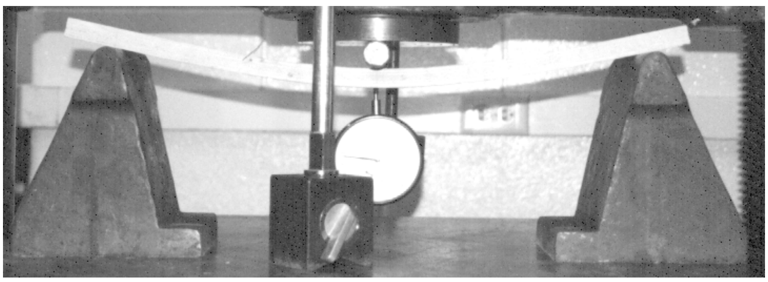


Fig. 5.2

Ductile failure – The plastic deformation in the structure that exhibits ductile behavior and is carried to the extreme so that it separates into two or more pieces as an example shown in Fig. 5.3.

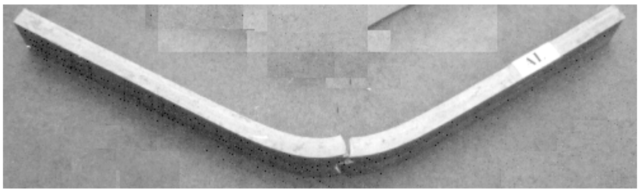


Fig. 5.3

Brittle failure – The elastic deformation in the structure that exhibits ductile behavior and is carried to the extreme so that it separates into two or more pieces as an example shown in Fig. 5.4.

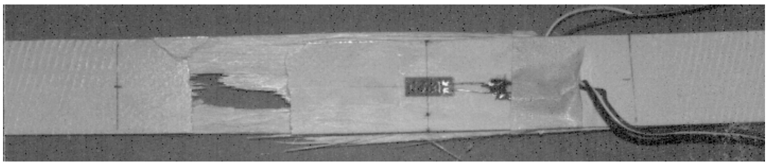


Fig. 5.4

Fatigue failure – The separation of a structure into two or more pieces or a certain size of crack initiation as a result of fatigue load or deformation for a period of time.

Low cycle fatigue: fatigue life < 10^5 cycles

High cycle fatigue: fatigue life $\geq 10^5$ cycles

Thermal fatigue

Sonic fatigue

Buckling failure – The deflection of a structure suddenly increased greatly with only a slight change in load. The buckled part is no longer capable of performing its intended function as an example shown in Fig. 5.5.

Fig. 5.5

Creep failure – The plastic deformation in a structure under the influence of stress or temperature over a period of time and becoming so large enough to interfere with the ability of the structure to satisfactorily performing its intended function.

5.3 Failure Criteria

The stress analysis itself can not be able to predict the failure of a structure. To know about how high stress can a structure be sustained or how high the strength of the structure, a failure criteria is needed.

The criteria discussed in this section will be focused on the failure due to static loads such as force-induced failure, yielding, ductile, and brittle failure. **Once the state of stresses at a critical point on a structure is determined, the principal stresses can be computed, and the failure criteria can be used.**

5.3.1 Maximum principal normal stress fracture criterion

Experimental observations show that brittle isotropic materials such as cast iron tend to fail suddenly by fracture without yielding.

Failure will occur when the maximum principal normal stress become equal to or exceed the maximum normal stress in a simple tension (or compression) test using a specimen of the same material.

Mathematically, if the material is subjected to plane stress the failure will occur when

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} \geq \sigma_{ult}$$

where σ_1, σ_2 are the principal normal stresses

σ_{ult} = ultimate tensile (or compressive) strength obtained from the tension test

Thus, we have

$$\left|\pm\frac{\sigma_1}{\sigma_{ult}}\right| = 1 \qquad \text{and} \qquad \left|\pm\frac{\sigma_2}{\sigma_{ult}}\right| = 1$$

A plot of these equations is in the rectangular shape and is shown in Fig. 5.6.

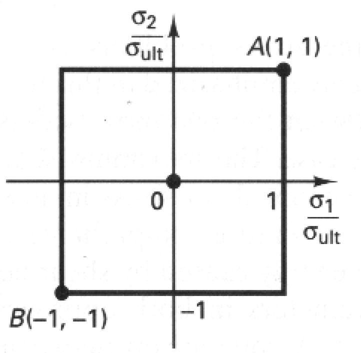


Fig. 5.6

Any stress falling within the rectangular indicates that the material behave elastically. Points on the hexagon indicate that the material is failing by separation or fracture.

5.3.2 Maximum shear stress yield criterion

Experimental evidence indicates that, in ductile isotropic material such as mild steel, slip occurs during yielding along critical oriented planes. This suggests that the maximum shearing stress play an important role in the failure of the ductile materials.

Failure will occur when the magnitude of the absolute maximum principal shear stress becomes equal to or exceed the maximum shear stress in a simple tension test using a specimen of the same material.

Mathematically, if the material is subjected to plane stress the failure will occur when

$$\tau_{\substack{abs\\ \max}} = \left|\frac{\sigma_1 - \sigma_2}{2}\right| \geq \left|\tau_y\right|$$

where $\tau_{\substack{abs\\ \max}}$ = absolute maximum principal shear stress

τ_y = maximum shear strength obtained from the tension test $= \dfrac{\sigma_y - 0}{2} = \dfrac{\sigma_y}{2}$.

Thus, $$\left|\sigma_1 - \sigma_2\right| \geq \left|\sigma_y\right|$$

$$\left|\pm\left[\frac{\sigma_1}{\sigma_y} - \frac{\sigma_2}{\sigma_y}\right]\right| = 1 \qquad \text{if } \sigma_1 \text{ and } \sigma_2 \text{ have the opposite signs}$$

$$\left.\begin{array}{l} \left|\pm\left[\dfrac{\sigma_1}{\sigma_y}\right]\right| = 1 \\ \left|\pm\left[\dfrac{\sigma_2}{\sigma_y}\right]\right| = 1 \end{array}\right\} \qquad \text{if } \sigma_1 \text{ and } \sigma_2 \text{ have the same signs}$$

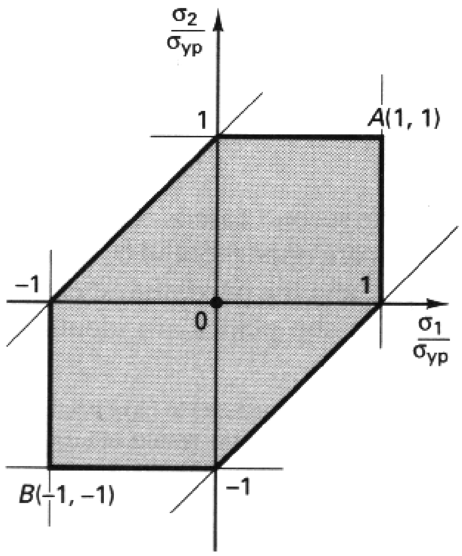


Fig. 5.7

A plot of these equations is in the hexagonal shape and is shown in Fig. 5.7. Any stress falling within the hexagon indicates that the material behave elastically. Points on the hexagon indicate that the material is yielding.

5.3.3 Maximum principal normal strain fracture criterion

Failure will occur when the maximum principal normal strain become equal to or exceed the ultimate strain in a simple tension (or compression) test using a specimen of the same material.

This criterion is an improvement over the maximum principal stress criterion, but it does not reliably predict failure by yielding. In practice, this criterion is rarely used excepting in the design of thick-walled cylinder. Mathematically, if the material is subjected to plane stress the failure will occur when

$$|\varepsilon_1| \geq |\varepsilon_{ult}|$$

or

$$|\varepsilon_2| \geq |\varepsilon_{ult}|$$

where $\varepsilon_1, \varepsilon_2$ are the principal strains which are $\varepsilon_1 = \dfrac{\sigma_1}{E} - \nu\dfrac{\sigma_2}{E}$ and $\varepsilon_2 = \dfrac{\sigma_2}{E} - \nu\dfrac{\sigma_1}{E}$.

$\varepsilon_{ult} = \dfrac{\sigma_{ult}}{E} =$ maximum tensile (or compressive) strain obtained from the tension test

Thus,

$$|\pm(\sigma_1 - \nu\sigma_2)| = \sigma_{ult}$$

$$|\pm(\sigma_2 - \nu\sigma_1)| = \sigma_{ult}$$

A plot of these equations is shown in Fig. 5.8.

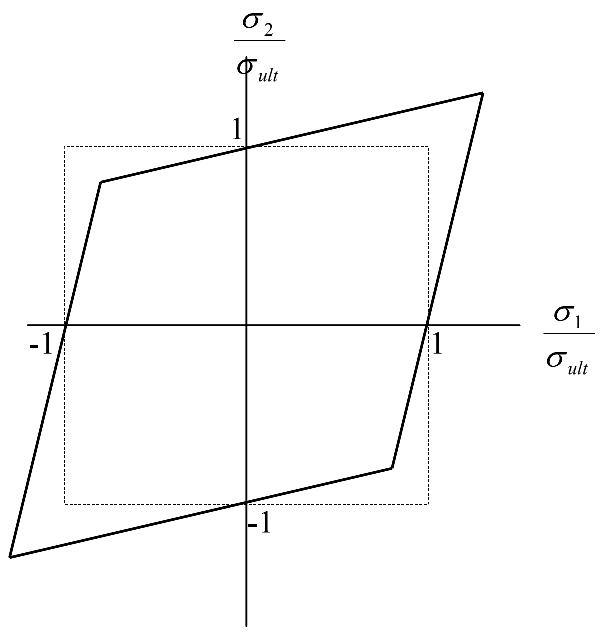


Fig. 5.8

5.3.4 Maximum distortion energy yield criterion

Failure will occur when the distortion energy density become equal to or exceed the distortion energy density at failure in a simple tension test using a specimen of the same material.

The total strain energy density U_o can be divided into two parts:

1. Due to solely volume change or dilation, $U_{o,v}$.
2. Due to solely change in shape or distortion, $U_{o,d}$.

$$U_o = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$$

Since the strain-stress relations in the form of principal strains and principal stresses are $\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)$, $\varepsilon_2 = \frac{1}{E}(\sigma_2 - \sigma_1 - \nu\sigma_3)$, and $\varepsilon_3 = \frac{1}{E}(\sigma_3 - \nu\sigma_2 - \nu\sigma_1)$, then, we can write the total strain energy density as

$$U_o = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

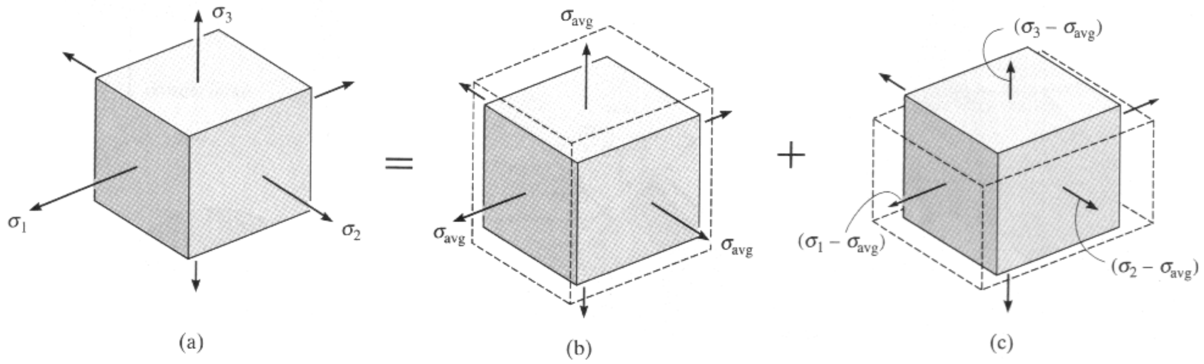


Fig. 5.9

From chapter 1, we have the *mean stress* or hydrostatic stress acting on a stress element which is the average of the principal stresses $\sigma_{avg.} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$. The corresponding hydrostatic strains can be determined as

$$\varepsilon_{avg.} = \frac{1}{E}(1 - 2\nu)\sigma_{avg.}$$

The dilation strain energy density,

$$U_{o,v} = 3\frac{1}{2}\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)\left(\frac{1-2\nu}{E}\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)$$

$$U_{o,v} = \frac{3}{2}\left(\frac{1-2\nu}{E}\right)\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)^2$$

The distortion strain energy density is $U_{o,d} = U_o - U_{o,v}$. Thus,

$$U_{o,d} = \frac{1}{2E}\left(\frac{1+\nu}{3}\right)\left((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right)$$

The distortion strain energy density in simple tension test at failure is

$$U_{o,dy} = \left(\frac{1+\nu}{3E}\right)\sigma_y^2$$

Mathematically, if the material is subjected to a general state of stresses the failure will occur when $U_{o,d} = U_{o,dy}$. Thus,

$$\frac{1}{2E}\left(\frac{1+\nu}{3}\right)\left((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right) = \left(\frac{1+\nu}{3E}\right)\sigma_y^2$$

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2$$

If the material is subjected to plane stress, the failure will occur when

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2)^2 + (-\sigma_1)^2 = 2\sigma_y^2$$

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_y^2$$

$$\left(\frac{\sigma_1}{\sigma_y}\right)^2 - \left(\frac{\sigma_1}{\sigma_y}\frac{\sigma_2}{\sigma_y}\right) + \left(\frac{\sigma_2}{\sigma_y}\right)^2 = 1$$

This is an equation of an ellipse. The plot of this equation is shown in Fig. 5.10. Any stress falling within the ellipse indicates that the material behave elastically. Points on the ellipse indicate that the material is yielding. Experimental investigations show that this criterion is best fitted for isotropic materials that fail by yielding or ductile rupture.

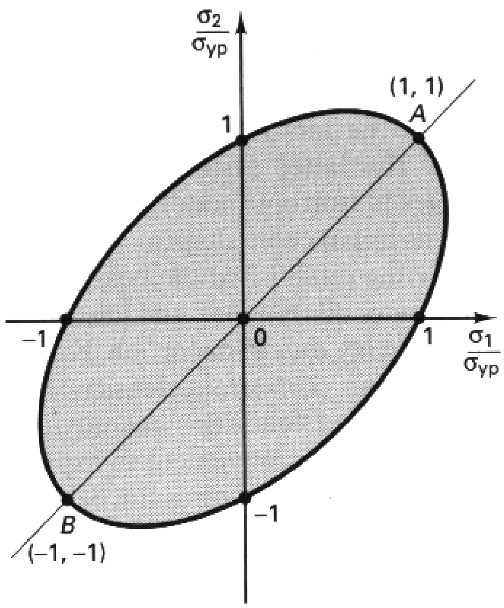


Fig. 5.10

5.3.5 Maximum octahedral shearing stress yield criterion

This failure criterion gives the same results as the maximum distortion energy criterion does. However, this criterion provides us to deal only with the stresses instead of the energy. From does the maximum distortion energy criterion, we have

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 = 2\sigma_y^2$$

From chapter 1, we have the octahedral shearing stress as

$$\tau_{oct}=\frac{1}{3}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2}$$

or

$$\tau_{oct}=\frac{1}{3}\left[(\sigma_x-\sigma_y)^2+(\sigma_x-\sigma_z)^2+(\sigma_y-\sigma_z)^2+6\tau_{xy}^2+6\tau_{xz}^2+6\tau_{yz}^2\right]^{1/2}$$

Mathematically, if the material is subjected to three-dimensional stress, the failure will occur when

$$\tau_{oct}=\frac{\sqrt{2}}{3}\sigma_y$$

Failure will occur when the maximum octahedral shearing stress become equal to or exceed the octahedral shearing stress at failure in a simple tension test using a specimen of the same material.

5.3.6 Coulomb-Mohr fracture criterion

In some brittle materials such as gray cast iron and concrete, the tension and compression properties are different. The failure of these materials should be predicted by using the Coulomb-Mohr criterion.

In the Coulomb-Mohr criterion, the fracture is hypothesized to occur on a given plane in the material when a critical combination of shear and normal stress acts on this plane. The simplest mathematical relation giving the critical combination of stresses is in the form of linear relationship. At fracture, we have

$$|\tau|+\mu\sigma=\tau_i$$

where τ and σ are the shearing stress and normal stress acting on the fracture plane, respectively, and μ and τ_i are constants for a given material. This equation forms a line on a plot of σ versus $|\tau|$ and the line has a slope of $-\mu$ and intercepts the τ axis at τ_i as shown in Fig. 5.11.

Consider a stress element subjected to the principal stress σ_1, σ_2, and σ_3. The Mohr's circle of the state of stresses can be drawn as shown in Fig. The failure is occurred if the largest of the three circles touch the line. Thus, the line represents a failure envelope for the Mohr's circle.

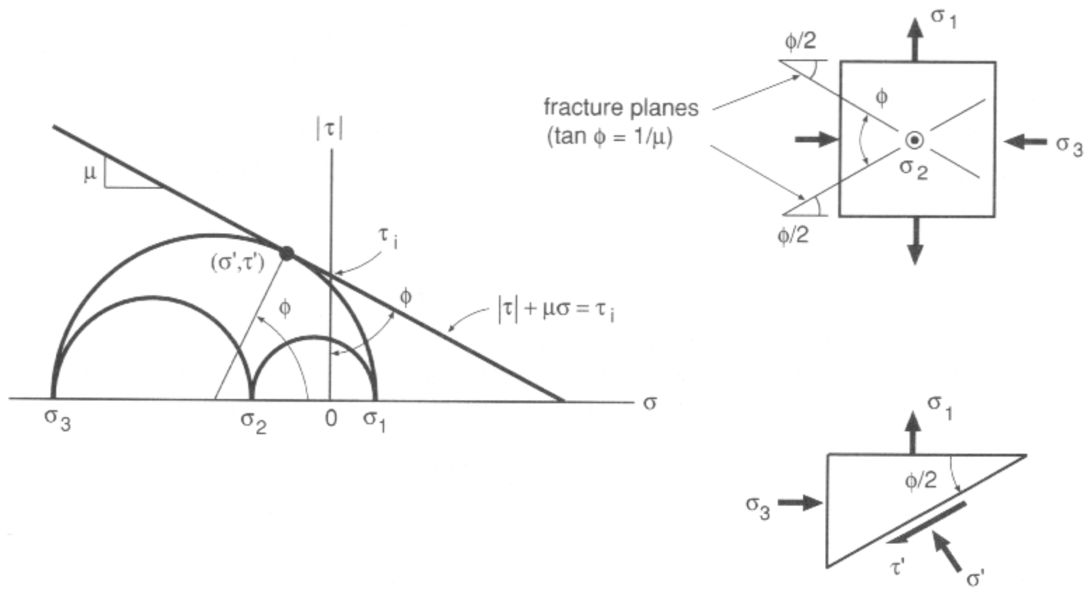


Fig. 5.11

As shown in the figure, the touching point has the coordinate (σ', τ') where

$$\sigma' = \frac{\sigma_1+\sigma_3}{2} + \left|\frac{\sigma_1-\sigma_3}{2}\right|\cos\phi$$

$$|\tau'| = \left|\frac{\sigma_1-\sigma_3}{2}\right|\sin\phi$$

Also, the failure planes are occurred where the maximum principal stress acts by a rotation $\phi/2$ in either direction and

$$\tan\phi = \frac{1}{\mu}$$

Substituting σ', $|\tau'|$, and $\tan\phi = \dfrac{1}{\mu}$ into the relation $|\tau| + \mu\sigma = \tau_i$, we have

$$|\sigma_1 - \sigma_3| + m(\sigma_1+\sigma_3) = 2\tau_u$$

where $m = \dfrac{\mu}{\sqrt{1+\mu^2}} = \cos\phi$ and $\tau_u = \dfrac{\tau_i}{\sqrt{1+\mu^2}} = \tau_i \sin\phi$.

For torsion test, we have $\sigma_1 = -\sigma_3 = \tau$ and $\sigma_2 = 0$. Thus,

$$|\tau+\tau| + m(\tau-\tau) = 2\tau_u$$

$$\tau = \tau_u$$

The τ_u is the pure shear necessary to causes fracture. The corresponding largest Mohr's circle and predicted fracture planes are show in Fig. 5.12.

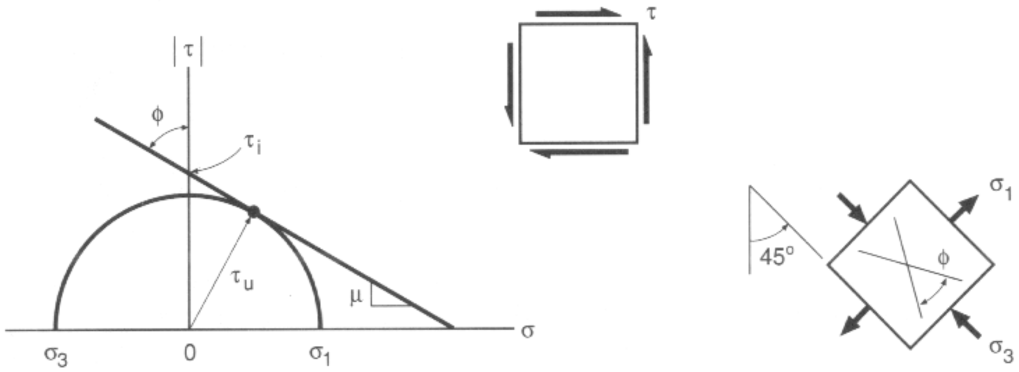


Fig. 5.12

For uniaxial tensile test, we have $\sigma_1 = \sigma_{ut}$ and $\sigma_2 = \sigma_3 = 0$. Thus,

$$|\sigma_{ut} - 0| + m(\sigma_{ut} + 0) = 2\tau_u$$

$$\sigma_{ut} = \frac{2\tau_u}{1+m}$$

For uniaxial compression test, we have $\sigma_1 = \sigma_2 = 0$ and $\sigma_3 = -\sigma_{uc}$. Thus,

$$|0 + \sigma_{uc}| + m(0 - \sigma_{uc}) = 2\tau_u$$

$$\sigma_{uc} = \frac{2\tau_u}{1-m}$$

It should be noted that σ_{uc} must have a negative value or $\sigma_{uc} = -\frac{2\tau_u}{1-m}$ since it is the compressive stress. The fracture planes predicted by the Coulomb-Mohr criterion for uniaxial tensile test and uniaxial compression test are shown in Fig 5.13.

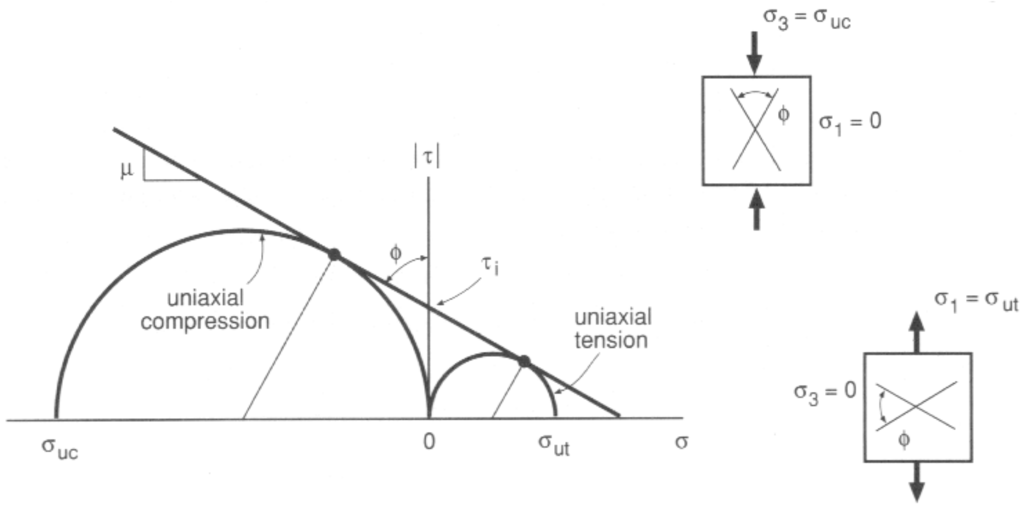


Fig 5.13

Eliminating τ_u , we have

$$\sigma_{uc} = -\frac{1+m}{1-m}\sigma_{ut}$$

Solving for m , we obtain

$$m = \frac{\sigma_{uc} + \sigma_{ut}}{\sigma_{uc} - \sigma_{ut}}$$

We can see that for a positive value of m, the strength in tension is predicted to be less than that in compression.

If the subscript for the principal stresses are assumed to be arbitrary assigned, then,

$$|\sigma_1 - \sigma_3| + m(\sigma_1 + \sigma_3) = 2\tau_u$$

$$|\sigma_2 - \sigma_3| + m(\sigma_2 + \sigma_3) = 2\tau_u$$

$$|\sigma_1 - \sigma_2| + m(\sigma_1 + \sigma_2) = 2\tau_u$$

For plane stress,

$$|\sigma_1| + m\sigma_1 = 2\tau_u$$

$$|\sigma_2| + m\sigma_2 = 2\tau_u$$

$$|\sigma_1 - \sigma_2| + m(\sigma_1 + \sigma_2) = 2\tau_u$$

The plot of this equation is shown in Fig. 5.14.

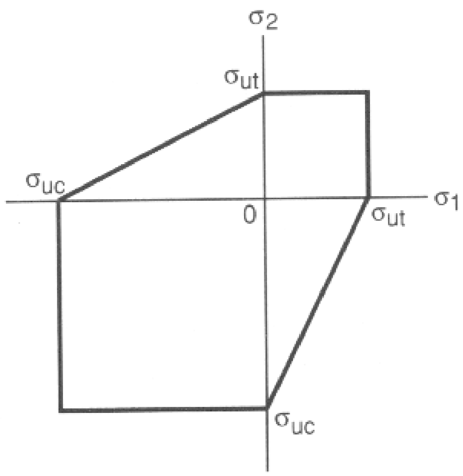


Fig. 5.14

It should be noted that the Coulomb-Mohr criterion with the constant $m = 0$ is equivalent to the maximum shear stress criterion and Fig. 5.14 will be the same as Fig. 5.7.

5.4 Comparison of the Failure Criteria

Fig. 5.15 shows the experimental results with the failure criteria presented before. It is concluded that

1. The maximum principal stress criterion is best fitted for isotropic material that is failed by the brittle fracture.
2. The maximum distortion energy criterion is best fitted for isotropic materials that fail by yielding or ductile rupture.

3. The maximum shearing stress criterion is almost as good as the maximum distortion energy criterion for isotropic materials that fail by yielding or ductile rupture.

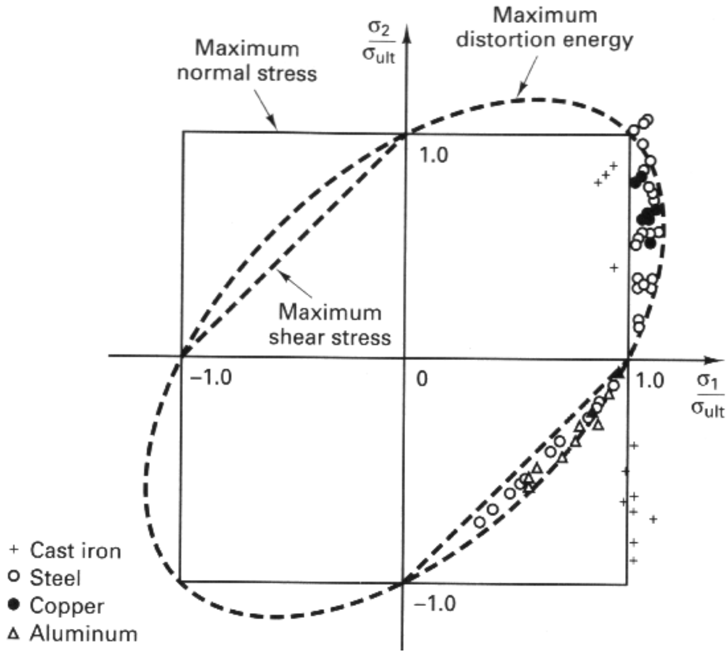


Fig. 5.15

Example 5-1

The stesolid shaft as shown in Fig. Ex 5-1a has a radius of 12.7 mm and is made of steel having $\sigma_y = 250\,\text{MPa}$. Determine if the loadings cause the shaft to fail according to the maximum shearing stress criterion and the maximum distortion energy criterion.

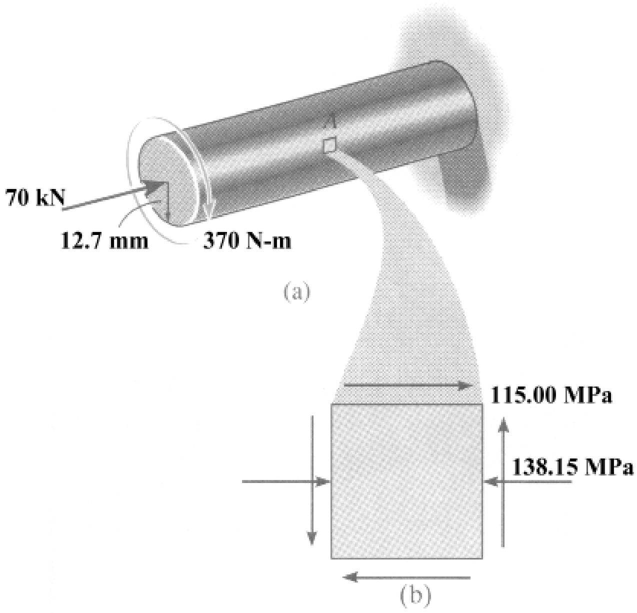


Fig. Ex5-1

Let the x - axis is in the longitudinal direction of the shaft. The averaged axial stress due to the axial force is

$$\sigma_x = \frac{70}{\pi(0.0127)^2} = 138.15\,\text{MPa}$$

The maximum shear stress caused by the torque is

$$\tau_{xy} = \frac{370(0.127)}{\dfrac{\pi(0.0127)^4}{2}} = 115.0\,\text{MPa}$$

The stress element at point A is as shown in Fig. Ex 5-1b.

The principal normal stresses due to the state of stresses as shown in Fig. Ex 5-1b are

$$\sigma_{\frac{1}{2}} = \frac{-138.15+0}{2} \pm \sqrt{\left(\frac{-138.15-0}{2}\right)^2 + 115.0^2}$$

$$\sigma_1 = 65.07\,\text{MPa}$$

$$\sigma_2 = -203.23\,\text{MPa}$$

Maximum shearing stress criterion

Since the principal normal stresses have the opposite sign,

$$\left| \pm\left[\frac{\sigma_1}{\sigma_y} - \frac{\sigma_2}{\sigma_y}\right] \right| \overset{?}{\leq} 1$$

$$\left\|\left[\frac{65.07}{250}-\frac{-203.23}{250}\right]\right\| = 1.073 > 1.0$$

Thus, the loadings cause the shaft to fail according to the maximum shearing stress criterion.

Maximum distortion energy criterion

$$\left(\frac{\sigma_1}{\sigma_y}\right)^2 - \left(\frac{\sigma_1}{\sigma_y}\frac{\sigma_2}{\sigma_y}\right) + \left(\frac{\sigma_2}{\sigma_y}\right)^2 \overset{?}{\le} 1$$

$$\left(\frac{65.07}{250}\right)^2 - \left(\frac{65.07}{250}\frac{-203.23}{250}\right) + \left(\frac{-203.23}{250}\right)^2 = 0.940 < 1.0$$

Thus, the loadings do not cause the shaft to fail according to the maximum distortion energy criterion.

Example 5-2

A circular cylindrical shaft is made of steel with $\sigma_Y = 700\,\text{MPa}$, $E = 200\,\text{GPa}$, and $\nu = 0.29$. The shaft is subjected to a static bending $M = 13.0\,\text{KN-m}$ and a static torque of $T = 30.0\,\text{kN-m}$ as shown in Fig. Ex 5-2. Using the factor of safety of $SF = 2.60$, determine the minimum diameter of the shaft based on the maximum octahedral shearing stress criterion (or equivalently the maximum distortion energy criterion) and the maximum shearing stress criterion.

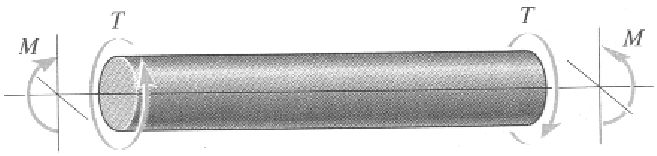


Fig. Ex 5-2

The shaft is subjected to a static bending moment $M = 13.0\,\text{KN-m}$ and a static torque of $T = 30.0\,\text{kN-m}$. However, due to the factor of safety of $SF = 2.60$, the moment and the torque must be increased by the factor. Thus, if we let the x- axis is in the longitudinal direction of the shaft, the stresses due to the loadings are

$$\sigma_x = SF\frac{Mc}{I} = \frac{32(SF)M}{\pi d^3}$$

$$\tau_{xy} = SF\frac{Tc}{J} = \frac{16(SF)T}{\pi d^3}$$

For the maximum octahedral shearing stress criterion,

$$\tau_{oct(\max)} = \frac{\sqrt{2}}{3}\sigma_Y$$

$$\frac{1}{3}\sqrt{(\sigma_x-\sigma_y)^2+(\sigma_x-\sigma_z)^2+(\sigma_y-\sigma_z)^2+6\tau_{xy}^2+6\tau_{xz}^2+6\tau_{yz}^2} = \frac{\sqrt{2}}{3}\sigma_Y$$

$$\frac{1}{3}\sqrt{2\sigma_x^2+6\tau_{xy}^2} = \frac{\sqrt{2}}{3}\sigma_Y$$

$$\sigma_Y = \sqrt{\sigma_x^2+3\tau_{xy}^2}$$

Substituting the stresses into the obtained equation, we get

$$\sigma_Y = \frac{16(SF)}{\pi d^3}\sqrt{4M^2+3T^2}$$

or

$$d_{\min} = \left[\frac{16(SF)}{\pi\sigma_Y}\sqrt{4M^2+3T^2}\right]^{1/3}$$

$$d_{\min} = 103\,\text{mm}$$

For the maximum shearing stress criterion,

$$\tau_{\max} = \frac{\sigma_Y}{2}$$

$$\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_Y}{2}$$

$$\frac{1}{2}\sqrt{\sigma_x{}^2 + 4\tau_{xy}^2} = \frac{\sigma_Y}{2}$$

Substituting the stresses into the obtained equation and rearranging the term, we get

$$d_{\min} = \left[\frac{32(SF)}{\pi\sigma_Y}\sqrt{M^2 + T^2}\right]^{1/3}$$

$$d_{\min} = 107 \text{ mm}$$

Chapter 6

Introduction to Fracture Mechanics

6.1 Introduction

Traditionally, the structural design approaches are based on the concept that the structures must have enough strength, stiffness, and stability to resist the loads.

For the strength criteria, the applied stress must be less than the yielding or ultimate strength of the material. However, when a crack is occurred in a component of the structure, it can cause the failure (in the form of fracture) at stresses well below the material's yielding strength. In this case, a special methodology called fracture mechanics can be used in design to minimize the possibility of failure.

Fracture mechanics is important in engineering design since cracks and crack-like flaw occur more frequently than we might expect. For example, the periodic inspections of large commercial aircraft frequently reveal cracks that must be repaired. Also, they are commonly occurred in ship structures, in bridge structures and in pressure vessel and piping.

The ability of a given material to resist a crack depends principally on the toughness of the material. Generally, fracture toughness in some metals such as steel increase with temperature as shown in Fig. 6.1. Also, there is an especially abrupt change in toughness over a relative small temperature range such as -50^oC for A469 steel. The temperature at this point is called the transition temperature. Thus, the fracture of the steel can be promoted by the temperature below the transition temperature.

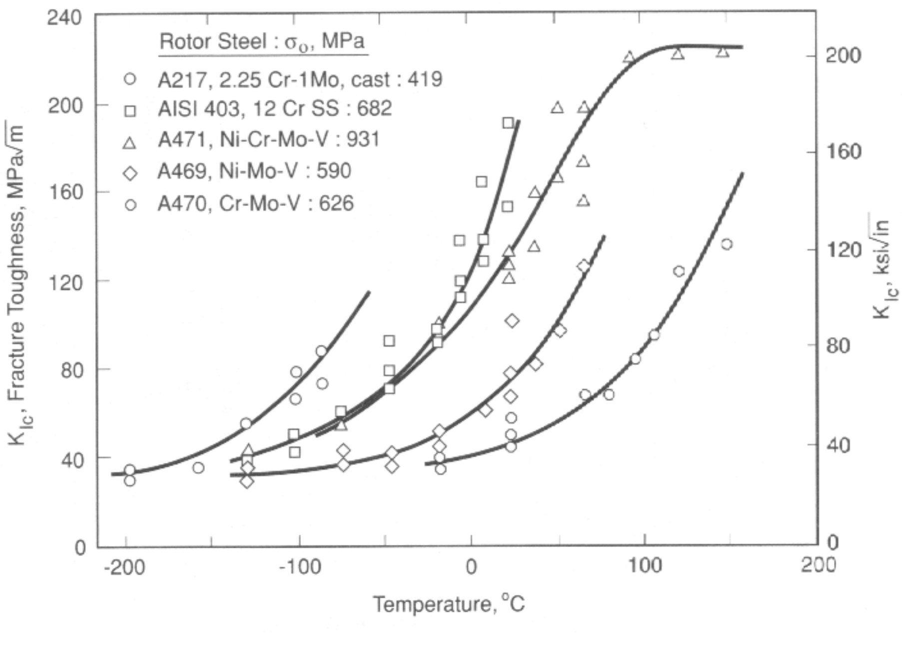


Fig. 6.1

Generally, fracture process can be categorized into three stages.

1. Crack initiation – micromechanics and dislocation theory

2. Crack extension – slow crack growth
3. Fast crack propagation

Thus, in fracture mechanics, a preexistent crack is assumed. Fracture mechanics is used to study the growth behavior of crack and residual strength of cracked structures, and to evaluate the life.

The growth of a crack and its corresponding stress can be shown as in Fig. 6.2.

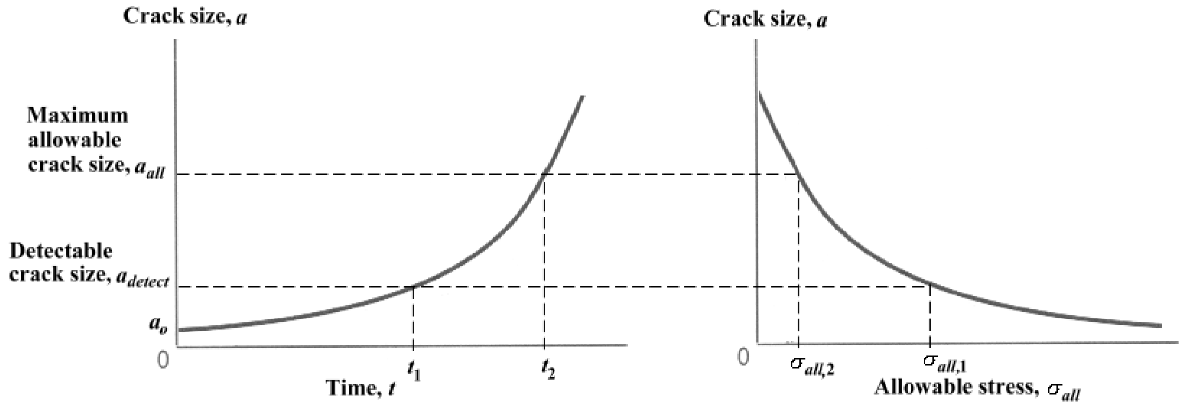


Fig. 6.2

The following questions are important in designing a structure using fracture mechanics. σ

1. What is the maximum permissible crack size?
2. What is the residual strength as a function of crack size?
3. How long does it take from the maximum detectable crack size to the maximum permissible crack size?
4. During the period available for crack detection, how often should the structure be inspected for crack?

6.2 Fracture Modes

Once a crack has been initiated, subsequent crack propagation may occur in several ways depending on the relative displacement of the particles in the two faces (surfaces) of the crack. There are three fundamental modes of fracture acting on the crack surface displacement as shown in Fig. 6.3.

1. Opening mode (Mode I) - The stress acts perpendicular to the crack growth direction and the crack growth plane. The crack surfaces move directly apart.
2. Shearing mode (Mode II) - The stress acts parallel to the crack growth direction and the crack growth plane. The crack surfaces move (slide) normal to the crack edge and remain in the plane of the crack.

3. Tearing mode (Mode III) - The stress acts perpendicular to the crack growth direction and parallel to the crack growth plane. The crack surfaces move parallel to the crack edge and remain in the plane of the crack.

The most general cases of crack surface displacements are obtained by superposition of these basic three modes.

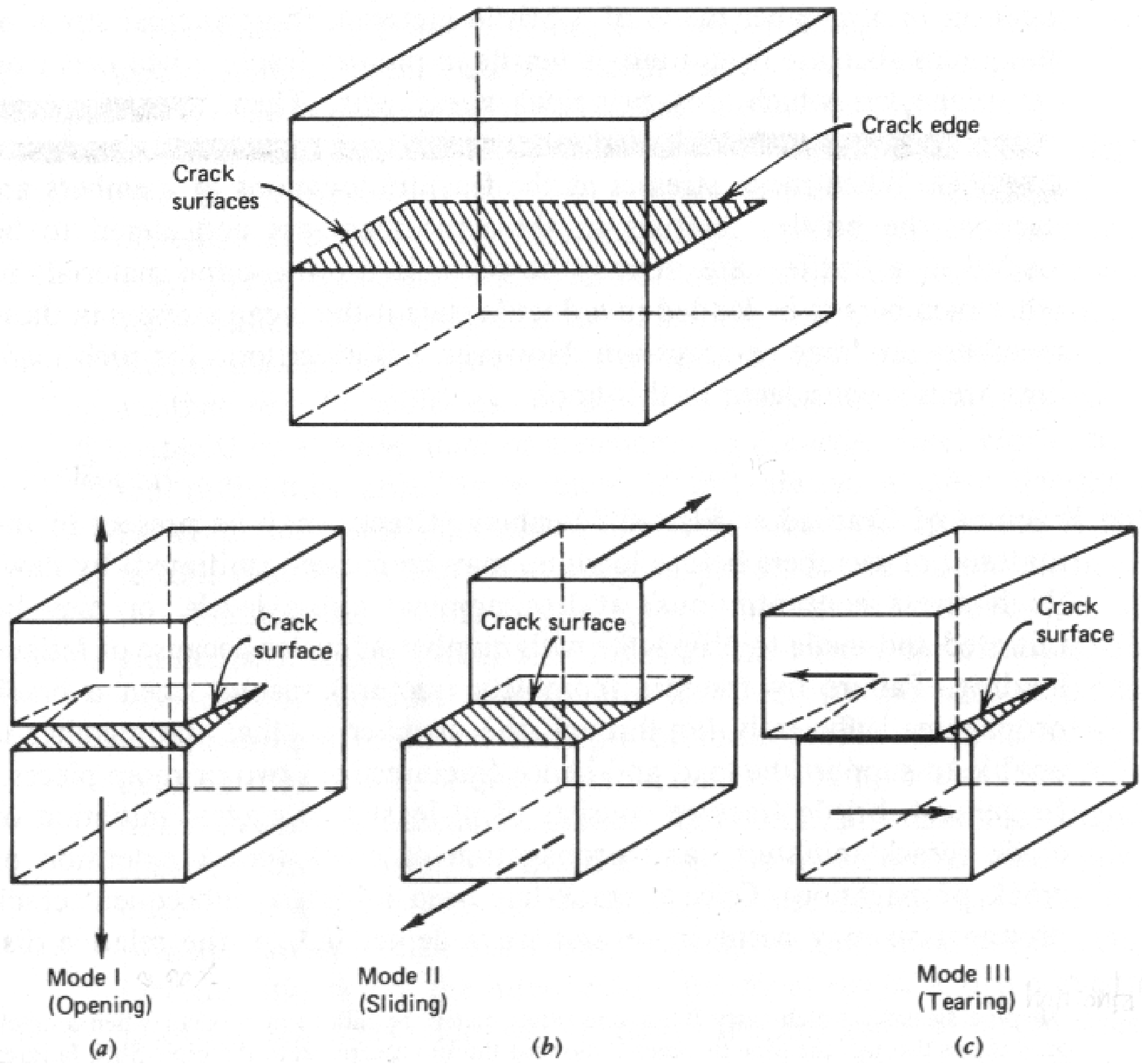


Fig. 6.3

6.3 Stress and Displacement Field at the Crack Tip

In 1950, Irwin showed that the local stresses near the crack tip, as the curvature at the crack tip goes to zero as shown in Fig. 6.4, are of the form

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta)$$

and we can see that $\sigma_{ij} \to \alpha$ as $r \to 0$. Thus, the stress field is a singular stress field with a singularity of $\sqrt{r}$. The term K is called the *stress intensity factor*, which defines the intensities or magnitudes of the singular stress around the crack tip. The expression of K depends on the fracture modes.

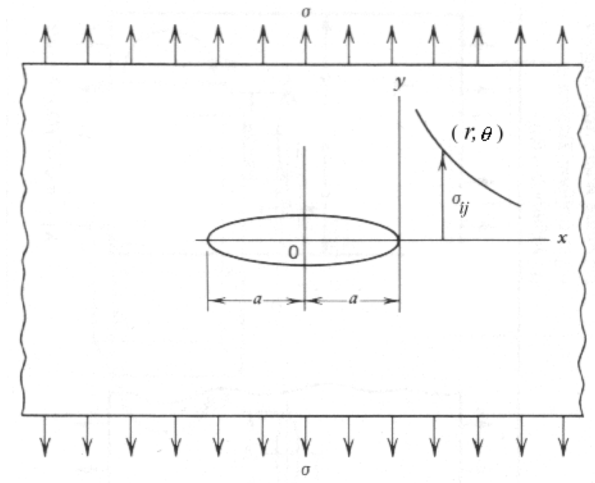


Fig. 6.4

6.3.1 Mode I

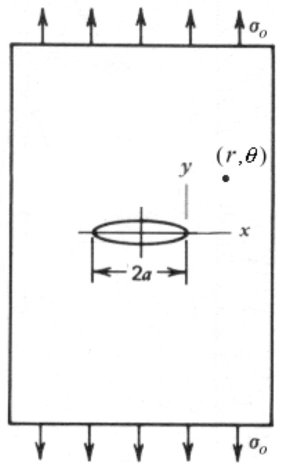


Fig. 6.5

The stresses at a point having a distance r and angle θ from the crack tip and for Mode I as shown in Fig. 6.5 are

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

Plane stress $\sigma_z = \tau_{yz} = \tau_{xz} = 0$

Plane strain $\sigma_z = \nu(\sigma_x + \sigma_y)$

$\tau_{yz} = \tau_{xz} = 0$

where $K_I = \lim_{r\to 0}\sigma_y\big|_{\theta=0}(\sqrt{2\pi r}) = \sigma_o\sqrt{\pi a}$

The displacements at a point having a distance r from the crack tip and angle θ with the x axis are

$$u_x = \frac{K_I}{8G}\sqrt{\frac{r}{2\pi}}\left[\cos\frac{\theta}{2}(2K-1) - \cos\frac{3\theta}{2}\right]$$

$$u_x(\theta) = u_x(-\theta)$$

$$u_y = \frac{K_I}{8G}\sqrt{\frac{r}{2\pi}}\left[\sin\frac{\theta}{2}(2K-1) - \sin\frac{3\theta}{2}\right]$$

$$u_y(\theta) = -u_y(-\theta)$$

where
$$K = \begin{cases} 3-4\nu & \text{plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

6.3.2 Mode II

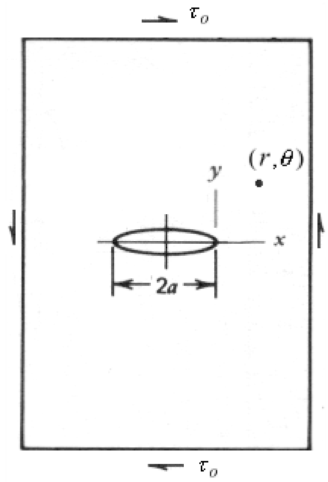


Fig. 6.6

The stresses at a point having a distance r from the crack tip and angle θ with the x axis for Mode II as shown in Fig. 6.6 are

$$\sigma_x = -\frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_y = \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

Plane stress $\quad \sigma_z = \tau_{yz} = \tau_{xz} = 0$

Plane strain $\quad \sigma_z = \nu(\sigma_x + \sigma_y)$

$$\tau_{yz} = \tau_{xz} = 0$$

where $\quad K_{II} = \tau_o\sqrt{\pi a}$

The displacements at a point having a distance r from the crack tip and angle θ with the x axis for Mode II are

$$u_x = \frac{K_{II}}{8G}\sqrt{\frac{2r}{\pi}}\left[(2K+3)\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right]$$

$$u_x(\theta) = -u_x(-\theta)$$

$$u_y = \frac{K_{II}}{8G}\sqrt{\frac{2r}{\pi}}\left[-(2K-3)\cos\frac{\theta}{2}-\cos\frac{3\theta}{2}\right]$$

$$u_y(\theta) = u_y(-\theta)$$

where $$K = \begin{cases} 3-4\nu & \text{plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

6.3.3 Mode III

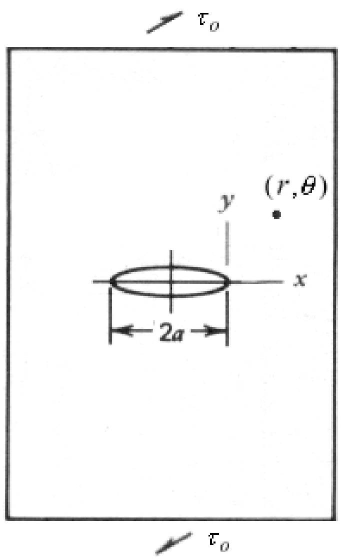


Fig. 6.7

The stresses at a point having a distance r from the crack tip and angle θ with the x axis for Mode III as shown in Fig. 6.7 are

$$\tau_{xz} = -\frac{K_{III}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}$$

$$\tau_{yz} = \frac{K_{III}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$$\sigma_z = \sigma_y = \sigma_z = \tau_{xy} = 0$$

where $$K_{III} = \tau_o\sqrt{\pi a}$$

The displacements at a point having a distance r from the crack tip and angle θ with the x axis for Mode III are

$$u_z = \frac{K_{III}}{G}\sqrt{\frac{2r}{\pi}}\sin\theta \quad \text{and} \quad u_x = u_y = 0$$

6.4 Stress Intensity Factor (SIF or K)

Stress intensity factors are needed to measure the intensity or magnitude of the singular stress field in the vicinity of an ideally sharp crack tip **in a linear elastic and isotropic material**. This approach is called *linear-elastic fracture mechanics* (LEFM). The factors do depend on loading condition, crack size, crack shape, and geometric boundaries. The general form of the stress intensity factors is given by

$$K = f.\sigma\sqrt{\pi a}$$

where σ = applied stress

a = effective crack length

f = correction factor. For infinity plate, $f = 1$.

Thus, stress intensity factor K has a unit in $\text{ksi}\sqrt{\text{in}}$ or $\text{MPa}\sqrt{\text{m}}$. The solutions of the stress intensity factors have been obtained for wide variety of problems and published in a handbook form. The followings are the typical solution for SIF:

6.4.a Center crack

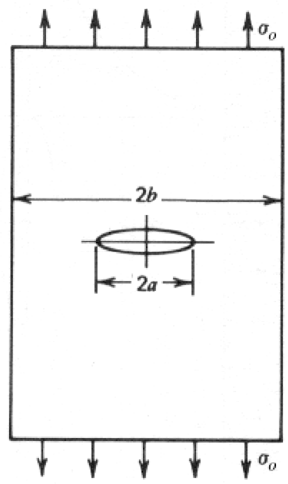


Fig. 6.8

$$K_I = \sigma_o\sqrt{\pi a}.f\left(\frac{a}{2b}\right)$$

$$f\left(\frac{a}{2b}\right) = \sqrt{\sec\left(\frac{\pi a}{2b}\right)}$$

6.4.b Double edge crack

$$K_I = \sigma_o\sqrt{\pi a}.f\left(\frac{a}{b}\right)$$

$$f\left(\frac{a}{b}\right)=1.12+0.203\frac{a}{b}-1.197\left(\frac{a}{b}\right)^2+1.930\left(\frac{a}{b}\right)^3$$

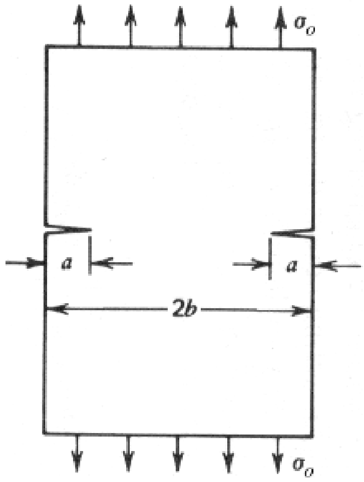


Fig. 6.9

6.4.c Single edge crack

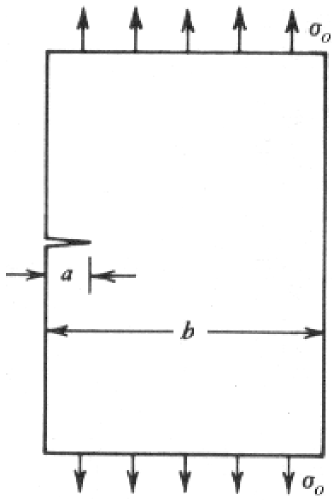


Fig. 6.10

$$K_I=\sigma_o\sqrt{\pi a}.f\left(\frac{a}{b}\right)$$

$$f\left(\frac{a}{b}\right)=1.12-0.231\frac{a}{b}+10.55\left(\frac{a}{b}\right)^2-21.72\left(\frac{a}{b}\right)^3+30.39\left(\frac{a}{b}\right)^4$$

6.4.d Crack under bending

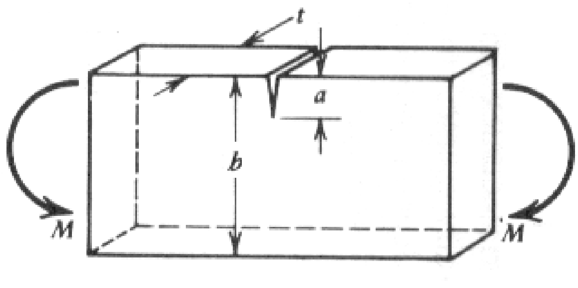


Fig. 6.11

$$K_I = \sigma_o \sqrt{\pi a}.f\left(\frac{a}{b}\right)$$

$$\sigma = \frac{6M}{b^2 t}$$

$$f\left(\frac{a}{b}\right) = 1.12 - 1.40\frac{a}{b} + 7.33\left(\frac{a}{b}\right)^2 - 13.08\left(\frac{a}{b}\right)^3 + 14\left(\frac{a}{b}\right)^4$$

6.5 Superposition of SIF

Stress intensity factor for combined loading can be obtained by the superposition method, that is, by adding the contribution to K from the individual load components. It is valid only for combination of the same mode of failure.

Consider an eccentric load applied at a distance e from a centerline of a member with a single edge crack as shown in Fig. 6.12. This eccentric load is statically equivalent to the combination of a centrally applied tension load and a bending moment.

The stress intensity factor for the centrally applied tension load is

$$K_I^a = \frac{P}{bt}\sqrt{\pi a}.f^a\left(\frac{a}{b}\right)$$

The stress intensity factor for the bending moment is

$$K_I^b = \frac{6M}{b^2 t}\sqrt{\pi a}.f^b\left(\frac{a}{b}\right)$$

Thus, the total stress intensity factor of this case is

$$K_I = K_I^a + K_I^b = \frac{P}{bt}\left(f^a\left(\frac{a}{b}\right) + \frac{6e}{b}f^b\left(\frac{a}{b}\right)\right)\sqrt{\pi a}$$

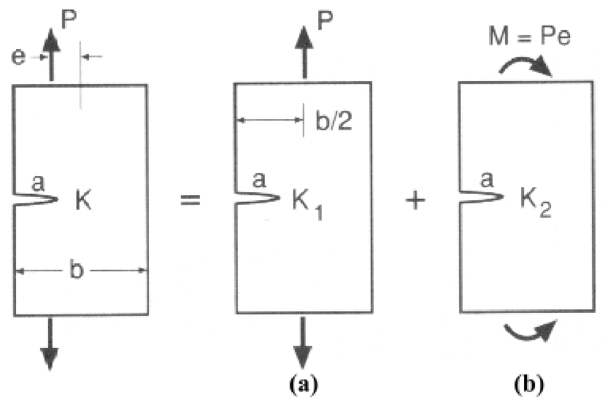


Fig. 6.12

6.6 Fracture Toughness (Critical SIF)

Fracture toughness K_{IC} is the critical value of the stress intensity factor K. If the stress intensity factor K occurred in a given material is less than the fracture toughness, the material will have ability to resist the crack without brittle fracture.

Fracture toughness is a material parameter, but it depends on both temperature and specimen thickness.

$$K_{IC} = \sigma_c \sqrt{\pi a}.f\left(\frac{a}{b}\right)$$

$$\sigma_c = \frac{K_{IC}}{\sqrt{\pi a}\, f\left(\frac{a}{b}\right)}$$

$$K_{IC} = \sigma \sqrt{\pi a_c}.f\left(\frac{a}{b}\right)$$

where a_c = critical crack size.

It should be noted that in order to ensure that the state of stress is plane strain for each of the cases in Section 6-4, the magnitudes of the crack half-length a and the thickness t should satisfy

$$a, t \geq 2.5\left(\frac{K_{IC}}{\sigma_y}\right)^2$$

Table 6-1 shows K_{IC} at the room temperature for several metals.

Table 6-1

Material	σ_u MPa	σ_y MPa	K_{IC} MPa$\sqrt{\text{m}}$	*Minimum Values for B, a, t* mm
		Alloy Steels		
A533B	—	500	175	306.0
2618 Ni Mo V	—	648	106	66.9
V1233 Ni Mo V	—	593	75	40.0
124 K 406 Cr Mo V	—	648	62	22.9
17-7PH	1289	1145	77	11.3
17-4PH	1331	1172	48	4.2
Ph 15-7Mo	1600	1413	50	3.1
AISI 4340	1827	1503	59	3.9
		Stainless Steel		
AISI 403	821	690	77	31.1
		Aluminum Alloys		
6061-T651	352	299	29	23.5
2219-T851	454	340	32	22.1
7075-T7351	470	392	31	15.6
7079-T651	569	502	26	6.7
2024-T851	488	444	23	6.7
		Titanium Alloys		
Ti-6Al-4Zr-2Sn-0.5Mo-0.5V	890	836	139	69.1
Ti-6Al-4V-2Sn	852	798	111	48.4
Ti-6.5Al-5Zr-1V	904	858	106	38.2
Ti-6Al-4Sn-1V	889	878	93	28.0
Ti-6Al-6V-2.5Sn	1176	1149	66	8.2

Example 6-1

Determine the stress intensity factor for the edge-cracked beam having the crack half-length a of $1.75\,\text{in.}$ when subjected to a moment of $100\,\text{kips - ft}$. It the beam was made of an extremely tough steel that has $\sigma_y = 195\,\text{ksi}$ and a K_{IC} of $150\,\text{ksi.}\sqrt{\text{in.}}$. The width of the beam is $4\,\text{in.}$ and the depth of the beam is $12\,\text{in.}$. If the moment applied to the beam was increased to $400\,\text{kips - ft}$, would this beam fail?

The flexural stress due to the moment $100\,\text{kips - ft}$ is

$$\sigma = \frac{Mc}{I} = \frac{M(b/2)}{(tb^3/12)} = \frac{6M}{tb^2} = \frac{6(100)(12)}{4(12^2)} = 12.5\,\text{ksi}$$

Since the crack half-length a of the beam is $1.75\,\text{in.}$,

$$\frac{a}{b} = \frac{1.75}{12} = 0.14583$$

$$f\left(\frac{a}{b}\right) = 1.12 - 1.40\frac{a}{b} + 7.33\left(\frac{a}{b}\right)^2 - 13.08\left(\frac{a}{b}\right)^3 + 14\left(\frac{a}{b}\right)^4 = 1.329$$

The stress intensity factor for the edge-cracked beam is

$$K_I = 12.5\sqrt{\pi(1.75)}(1.329) = 38.95\,\text{ksi.}\sqrt{\text{in.}}$$

The flexural stress due to the moment $400\,\text{kips - ft}$ is

$$\sigma == \frac{6(400)(12)}{4(12^2)} = 50\,\text{ksi}$$

$$K_I = 50\sqrt{\pi(1.75)}(1.329) = 155.80\,\text{ksi.}\sqrt{\text{in.}}$$

Since the stress intensity factor is larger than K_{IC} of $150\,\text{ksi.}\sqrt{\text{in.}}$ and the flexural stress is less than the yielding strength $\sigma_y = 195\,\text{ksi}$, the beam does fail by fracture.

Example 6-2

A tool as shown in Fig. Ex 6-2 is used to dig up old road beds before replacing them. Let the tool be made of AISI 4340. The dimensions of the tool are $d = 250\,\text{mm}$, $b = 60\,\text{mm}$, and the width $t = 25\,\text{mm}$. Determine the magnitude of the fracture load P for the crack length of $a = 5\,\text{mm}$.

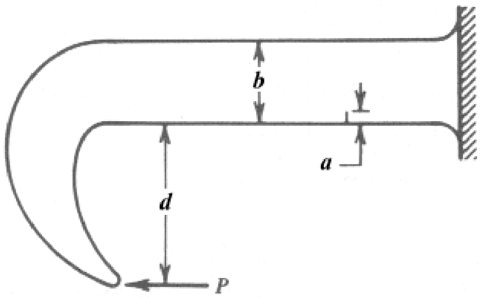


Fig. Ex 6-2

The crack half-length a and the thickness t satisfy the condition

$$a, t \geq 2.5\left(\frac{K_{IC}}{\sigma_y}\right) = 2.5\left(\frac{59}{1503}\right)^2 (10^3\ \text{mm/m}) = 3.9\,\text{mm}$$

Thus, we can consider the problem as a plane strain problem.

At the crack section, the tool is subjected to combine axial load and bending. Assuming the behavior of the tool is in the linear-elastic range and by using the superposition method. Thus,

$$K_I = K_I^a + K_I^b$$

For the crack length of $a = 5\,\text{mm}$, $a/b = 5/60 = 0.0833$.

$$f^a\left(\frac{a}{b}\right) = 1.12 - 0.231\frac{a}{b} + 10.55\left(\frac{a}{b}\right)^2 - 21.72\left(\frac{a}{b}\right)^3 + 30.39\left(\frac{a}{b}\right)^4 = 1.163$$

$$f^b\left(\frac{a}{b}\right) = 1.12 - 1.40\frac{a}{b} + 7.33\left(\frac{a}{b}\right)^2 - 13.08\left(\frac{a}{b}\right)^3 + 14\left(\frac{a}{b}\right)^4 = 1.047$$

$$K_{IC} = \left(\frac{P}{bt} f^a\left(\frac{a}{b}\right) + \frac{6M}{b^2 t} f^b\left(\frac{a}{b}\right)\right)\sqrt{\pi a}$$

$$59\sqrt{1000} = \left(\frac{P}{25(60)}(1.163) + \frac{6(280P)}{(60^2)25}(1.047)\right)\sqrt{\pi(5)}$$

$$P = 23.17\,\text{kN}$$

The total maximum stress is

$$\sigma = \frac{P}{bt} + \frac{6M}{b^2 t} = \frac{23.17}{0.025(0.060)} + \frac{6(0.280)23.17}{0.060^2(0.025)} = 448\,\text{MPa} < \sigma_y = 1503\,\text{MPa}$$

which is in accordance with the assumption of linear elastic behavior.

6.7 Strain Energy Release Rate and Its Equivalent to SIF

Consider a cracked member under a Mode I as shown in Fig. 6.13a. Assume that the behavior of the material is linear elastic under the action of load P. As a result of the elastic deformation, the strain energy stored in the member as shown in Fig. 6.13a is

$$U = \frac{1}{2} P v$$

where v is the displacement at the loading point.

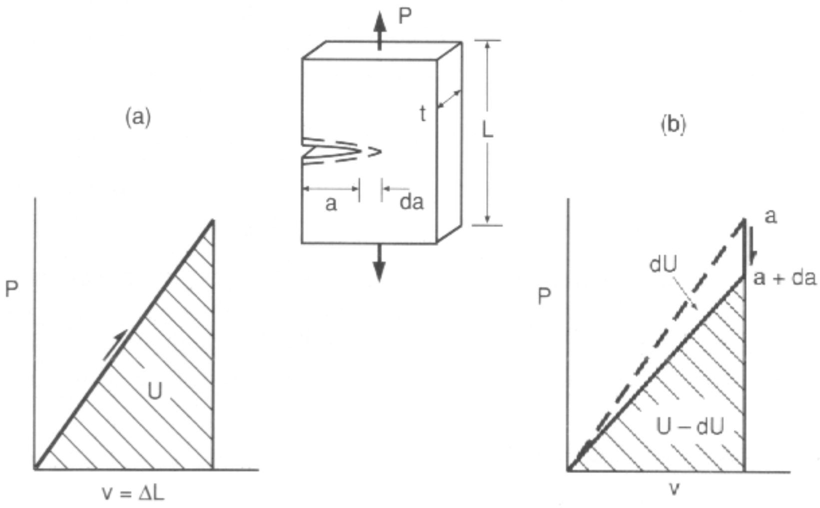


Fig. 6.13

If the crack moves ahead by a small distance da, while the displacement is held constant, the stiffness of the member decreases as shown in Fig. 6.13b. This results in the decreasing in the strain energy by the amount of dU, that is, U decreases due to a release of this amount of energy.

The strain energy release rate (G) is defined as the rate of change of strain energy with increase in crack area .

$$G = -\frac{\partial U}{\partial A} = -\frac{1}{t}\frac{\partial U}{\partial a}$$

where t = thickness of the plate. Since G has a unit in $\frac{\text{lb}-\text{in}}{\text{in}^2} = \frac{\text{lb}}{\text{in}}$, G is sometimes considered as a crack driving force.

For plane stress,

$$G_I = \frac{K_I^2}{E}$$

$$G_{II} = \frac{K_{II}^2}{E}$$

$$G_{III} = \frac{1+\nu}{E} K_{III}^2$$

For plane strain,

$$G_I = \frac{1-\nu^2}{E} K_I^2$$

$$G_{II} = \frac{1-\nu^2}{E} K_{II}^2$$

$$G_{III} = \frac{1+\nu}{E} K_{III}^2$$

6.8 Plastic Zone Size

Irwin has shown that the local stress at the crack tip as shown in Fig 6.14 is in the form of

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}(\theta)$$

This means that at the crack tip or $r \to 0$, the local stress is infinite or $\sigma_{ij} \to \alpha$. However, real materials can not support these theoretical infinite stresses. Thus, upon loading, the crack tip becomes blunted and a region of yielding or microcracking forms. This region of yielding is called *plastic zone*.

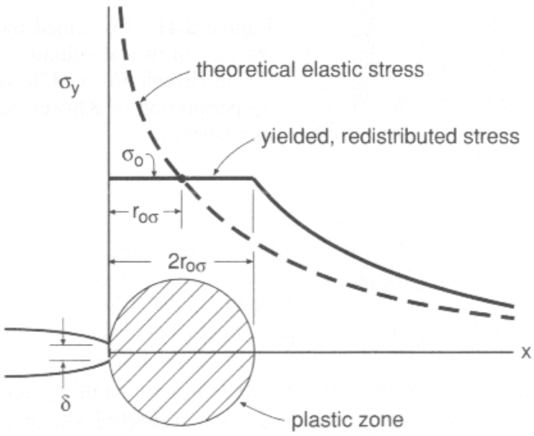


Fig. 6.14

For any cases of Mode I loading, the stresses near the crack tip are

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

Plane stress $\qquad \sigma_z = \tau_{yz} = \tau_{xz} = 0$

Plane strain $\qquad \sigma_z = \nu(\sigma_x + \sigma_y)$ and $\tau_{yz} = \tau_{xz} = 0$

where $$K_I = \sigma_o \sqrt{\pi a}$$

For plane stress, the state of stress at the plane of the crack where the angle $\theta = 0^o$ is

$$\sigma_x = \sigma_y = \frac{K_I}{\sqrt{2\pi r}}$$

$$\sigma_z = \tau_{yz} = \tau_{xz} = \tau_{xy} = 0$$

Since all shear stress along the plane of the crack are zero, σ_x, σ_y, and σ_z are the principal normal stresses. The maximum shear stress and the maximum octahedral shearing stress criteria estimate the yielding at

$$\sigma_x = \sigma_y = \sigma_{ys}$$

where σ_{ys} is the yielding strength. Therefore, we obtain the radius of the plastic zone for the plane stress in the form of

$$r_y = \frac{1}{2\pi}\left[\frac{K_I}{\sigma_{ys}}\right]^2$$

For the plane strain, the radius of the plastic zone can be determined by using the equation

$$r_y = \frac{1}{6\pi}\left[\frac{K_I}{\sigma_{ys}}\right]^2$$

It should be noted that the radius of the plastic zone for the plane strain is smaller than one of the plane stress. This is due to the fact that the stress σ_z for the plane strain is nonzero, and this elevates the value of $\sigma_x = \sigma_y$ necessary to cause yielding, in turn decreasing the plastic zone size relative to that for plane stress.

Thus, for different materials, the one having a lower σ_{ys} will have a larger r_y. The plastic zone size for plane stress condition is larger than that of the plane strain condition.

For cyclic loading, the cyclic plastic zone size can be determined by

For plane stress,

$$r_y = \frac{1}{8\pi}\left[\frac{K_I}{\sigma_{ys}}\right]^2$$

For plane strain,

$$r_y = \frac{1}{24\pi}\left[\frac{K_I}{\sigma_{ys}}\right]^2$$

Hence, we can see that $r_y\big|_{cyclic} < r_y\big|_{static}$.

Example 6-3

A large plate made of 4140 steel ($\sigma_{ys} = 90\,\text{ksi}$) containing a $0.2\,\text{in.}$ center crack is subjected to a tensile stress of $\sigma_o = 30\,\text{ksi}$.

a.) Determine the plastic zone size.

This problem is a plane stress problem. The plastic zone size can be determined by using the equation

$$r_y = \frac{1}{2\pi}\left[\frac{K_I}{\sigma_{ys}}\right]^2$$

The stress intensity factor,

$$K_I = \sigma_o\sqrt{\pi a}.f\left(\frac{a}{2b}\right)$$

$$f\left(\frac{a}{2b}\right) = \sqrt{\sec\left(\frac{\pi a}{2b}\right)} = 1.0$$

$$K_I = 30\sqrt{\pi(0.2)/2}.(1) = 16.81\,\text{ksi}.\sqrt{\text{in}}$$

Thus, the plastic zone size is $r_y = 5.56(10^{-3})$ in.

b.) Are the LEFM's assumptions violated?

The assumptions remain valid since the plastic zone size is small relative to the crack size and cracked body.

c.) If the yielding strength of the material is reduced by a factor of 2.0, cal culate the plastic zone size. Are the LEFM's assumptions violated

$$r_y = \frac{1}{2\pi}\left[\frac{16.81}{45}\right]^2 = 0.022\,\text{in.}$$

The assumptions are violated since the plastic zone size is quite large compared to the crack size (about 22% of the half crack size).

Chapter 7

Fatigue

7.1 Introduction

Structural members and mechanical parts are often found to have failed under the action of repeated or fluctuating stresses called fatigue failure. The actual maximum repeated stresses were well below the ultimate strength of the material and quite frequently evens below the yield strength. Typical fatigue failures do not involve macroscopic plastic deformation.

Fatigue failure often begins with a small crack. The initial crack is so minute that it can not be detected by the naked eyes and even by the X-ray method. The small cracks are usually developed at high stress gradient area. Once crack is developed, the stress gradient becomes larger and larger, and the crack progresses more rapidly.

At present, there are three major approaches to analyzing and designing against fatigue failure. They are stress-based approach, strain-based approach, and fracture mechanic approach.

Stress-based approach is based on the nominal stresses in the region of the component being analyzed. The nominal stress that can be resisted under cyclic loading is determined by considering mean stresses and by making adjustments for the effects of stress risers such as holes and fillet.

Strain-based approach involves more detailed analysis of the localized yielding that may occur at stress risers during cyclic loading.

Fracture mechanic approach is used to treat growing crack due to cyclic loading by using the method of fracture mechanics.

7.2 Nomenclature

Some practical applications involve cycling between maximum and minimum stress levels that are constant. This is called constant amplitude stressing as shown in Fig. 7.1 and 7.2.

The following nomenclatures for cyclic loading are important.

Mean stress is the average of the maximum stress and the minimum stress.

$$\sigma_m = \frac{\sigma_{\max} + \sigma_{\min}}{2}$$

Stress range is the difference between the maximum stress and the minimum stress.

$$\Delta\sigma = \sigma_{\max} - \sigma_{\min}$$

Stress amplitude is half of the stress range.

$$\sigma_a = \frac{\sigma_{\max} - \sigma_{\min}}{2}$$

Stress ratio is the ratio of the minimum stress over the maximum stress.

$$R = \frac{\sigma_{\min}}{\sigma_{\max}}$$

Amplitude ratio is the ratio of the stress amplitude over the mean amplitude

$$a = \frac{\sigma_a}{\sigma_m}$$

Cyclic loading needs two independent variables to specify. Some combinations that may be used are: σ_a and σ_m, $\sigma_{\max}$ and R, $\Delta\sigma$ and R, $\sigma_{\max}$ and $\sigma_{\min}$ and σ_a and a.

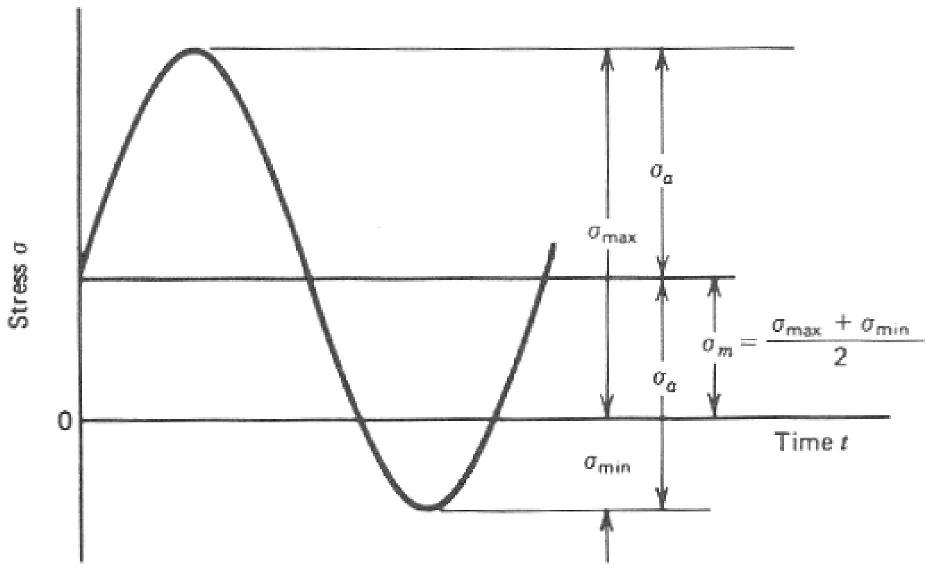


Fig. 7.1

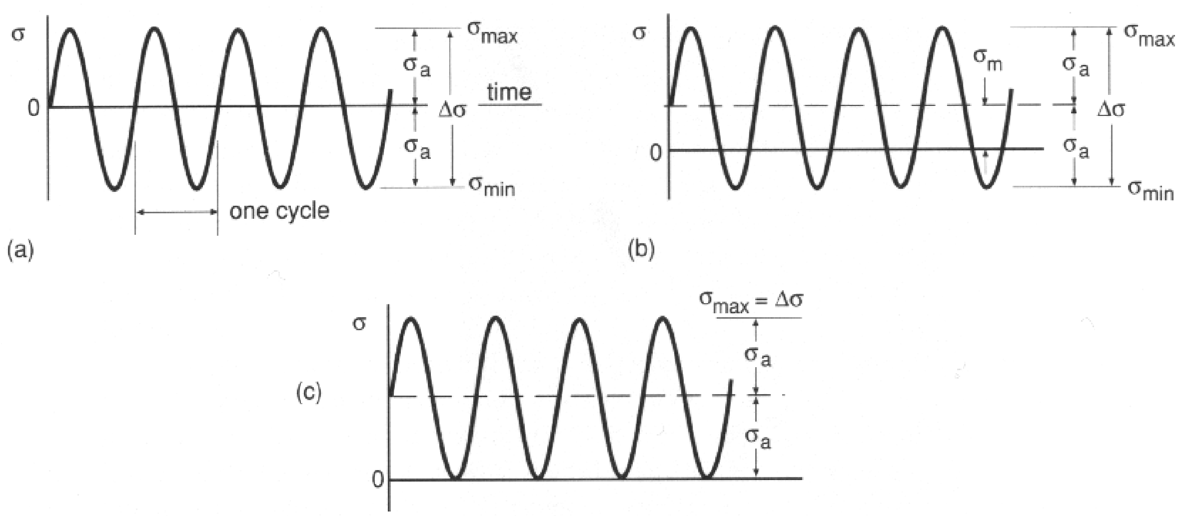


Fig. 7.2

Point stresses Versus Nominal Stresses

It is important to distinguish between the stress at a point, σ, and the nominal stress, S. Nominal stress, S, is equal to point stress, σ, only in certain situations.

For simple axial loading, the point stress, σ, is the same everywhere and so is equal to the nominal stress, $S = P / A$.

For bending, the nominal stress is usually calculated from the elastic bending equation, $S = Mc / I$. Hence, $\sigma = S$ at the edge of the bending member, with σ being less everywhere. However, if yielding occurs, the actual stress distribution becomes nonlinear, and σ at the edge of the bending member is no longer equal to S. Thus, in material testing, it is essential to distinguish S from σ when the yielding occurs.

For notched member, the nominal stress, S, is determined from the net area remaining after removal of the notch. Due to the stress raiser effect, the nominal stress, S, needs to be multiplied by a stress concentration factor, k_t. Thus, the peak stress at the notch σ is equal to $k_t S$.

7.3 Cyclic Stress-Strain Behavior of Metals

Consider a stress-strain response curve from a fatigue test of a metal specimen as shown by the solid line in Fig. 7.3. When the strain is increased from 0 to $\varepsilon_{\max}$, the stress is also increased from 0 to $\sigma_{\max}$ by following the dashed line of the stress-strain curve. Then, when we unload the specimen from the strain $\varepsilon_{\max}$ to $\varepsilon_{\min}$, the stress-strain curve follows the unloaded line, and the stress is decreased from $\sigma_{\max}$ to $\sigma_{\min}$. Finally, if we reload the specimen from $\varepsilon_{\min}$ to $\varepsilon_{\max}$, the stress is increased from $\sigma_{\min}$ to $\sigma_{\max}$ by following the reload curve. It can be seen that there is a loop occurred due to the unloading and reloading the specimen. This loop is called *hystereis loop*. It represents measurement of plastic deformation work done on the material. The area within the loop is the energy per unit volume dissipates during a cycle.

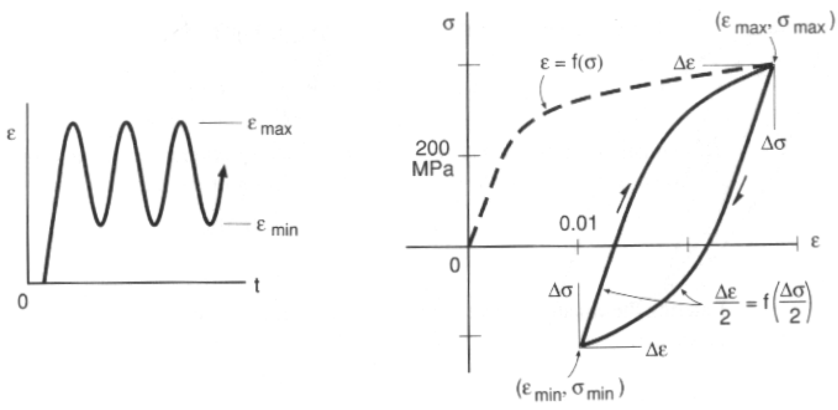


Fig. 7.3

Bauschinger Effect

Consider the monotonic tensile stress-strain curve as shown in Fig. 7.4a. If the specimen is loaded passed the yielding strength σ_{ot} to reach a maximum stress $\sigma_{\max}$ and the direction of straining is reversed, the stress-strain path that is followed differs from the initial monotonic one as illustrated in Fig. 7.4b. Yielding on unloading generally occurs prior to the stress reaching the yield strength σ_{oc} for monotonic compression, as at point A. This early yielding behavior is called the *Bauschinger effect*.

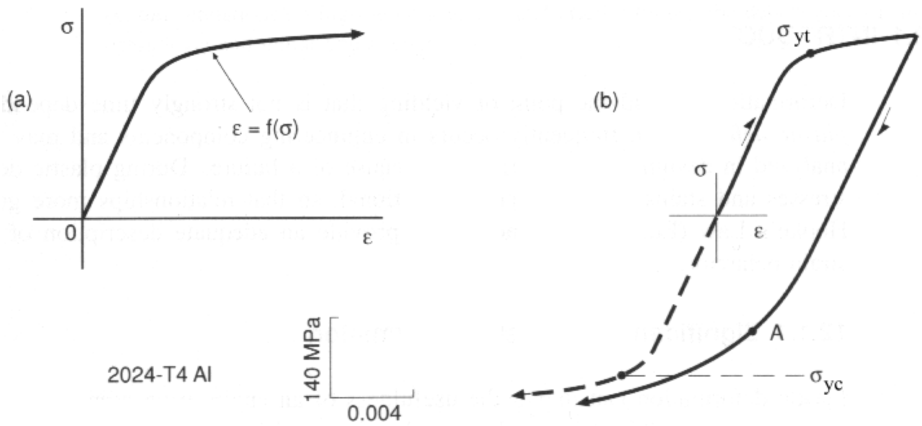


Fig. 7.4

The cyclic stress-strain response of metal is dramatically altered due to plastic strain. It depends on the initial conditions of the specimen such as quenching, tempering, and annealing, and its testing conditions.

Under the strain-controlled fatigue test, a metal specimen may exhibit the cyclic stress-strain response as following:

a.) Cyclically hardening

If the stress required to enforce the strain *increases* on subsequence reversals, the material undergoes cyclic hardening as shown in Fig. 7.5. In this case, the yield and ultimate strength of the material are increased. The example of the metal that exhibits this response is the annealed pure metal.

b.) Cyclically softening

If the stress required to enforce the strain *decreases* on subsequence reversals, the material undergoes cyclic softening as shown in Fig. 7.5. In this case, the yield and ultimate strength of the material are decreased. The example of the metal that exhibits this response is the cold worked pure metal.

c.) Cyclically stable

Through the cyclic hardening and softening, some intermediate strength levels are attained which represents a cyclically stable condition. The stable condition is usually reached in about 20-40% of total fatigue life.

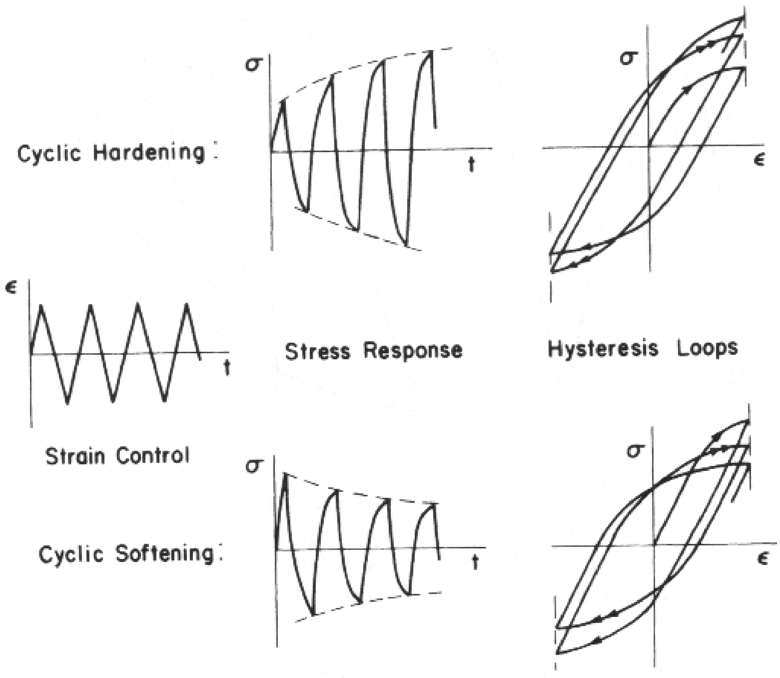


Fig. 7.5

d.) Mixed behavior

A material exhibits cyclic softening at the early stage of fatigue life and then cyclic hardening at the later stage of fatigue life.

7.3.1 Comparison of Metal Behavior between Monotonic and Cyclic Tests

Cyclic stress-strain curves for several engineering metals are compared with monotonic tension curves as shown in Fig. 7.6. When the cyclic curve is above the monotonic one, the material is one that cyclically hardens and when the cyclic curve is below the monotonic one, the material is one that cyclically softens. A mixed behavior may also occur, with crossing of the curves indicating softening at some strain levels and hardening at the others.

The following criteria were proposed by Manson.

If $\dfrac{S_{ult}}{S_{0.2\sigma_y}} > 1.4$, cyclically hardening.

If $\dfrac{S_{ult}}{S_{0.2\sigma_y}} < 1.2$, cyclically softening.

If $1.2 < \dfrac{S_{ult}}{S_{0.2\sigma_y}} < 1.4$, generally stable, or may hardening or softening.

The monotonic strain-hardening coefficient n is needed for predicting the material cyclic behavior. In general,

If $n > 0.20$, cyclically hardening.

If $n < 0.10$, cyclically softening.

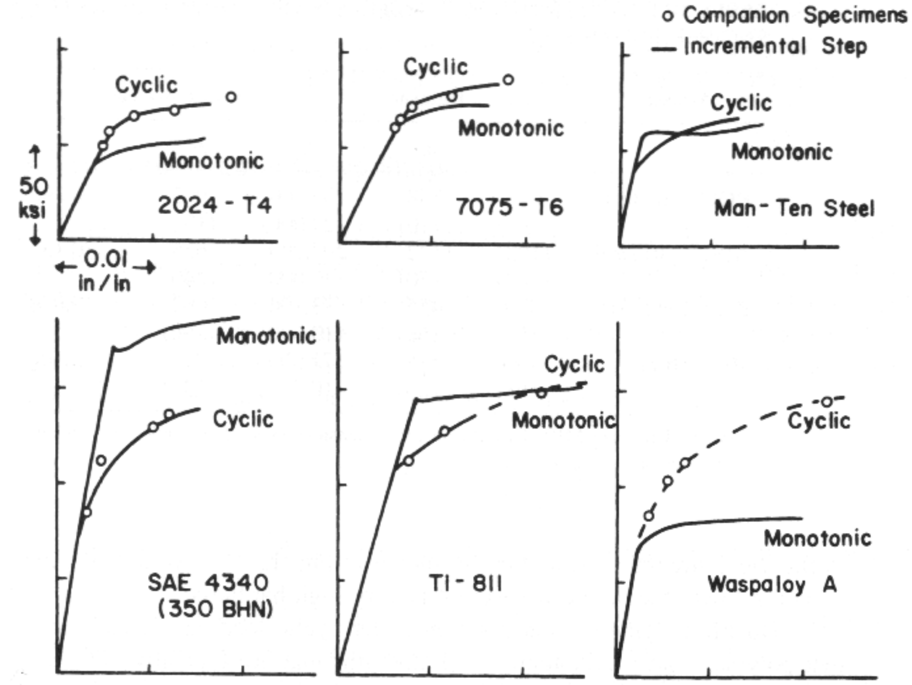


Fig. 7.6

7.4 Cyclic Stress-Strain Curve

There are several test methods that can be used to develop a cyclic stress-strain curve. For most metallic material, the controlled strain amplitude fatigue test will generate a stabilized hysterisis loop. The stress-strain curve is constructed by a sequence of the stabilized hysterisis loop.

a.) Companion sample method

The cyclic stress-strain curve is constructed by a set of test specimens at various strain levels as shown in Fig. 7.7. This method is time-consuming and requires a large number of tests.

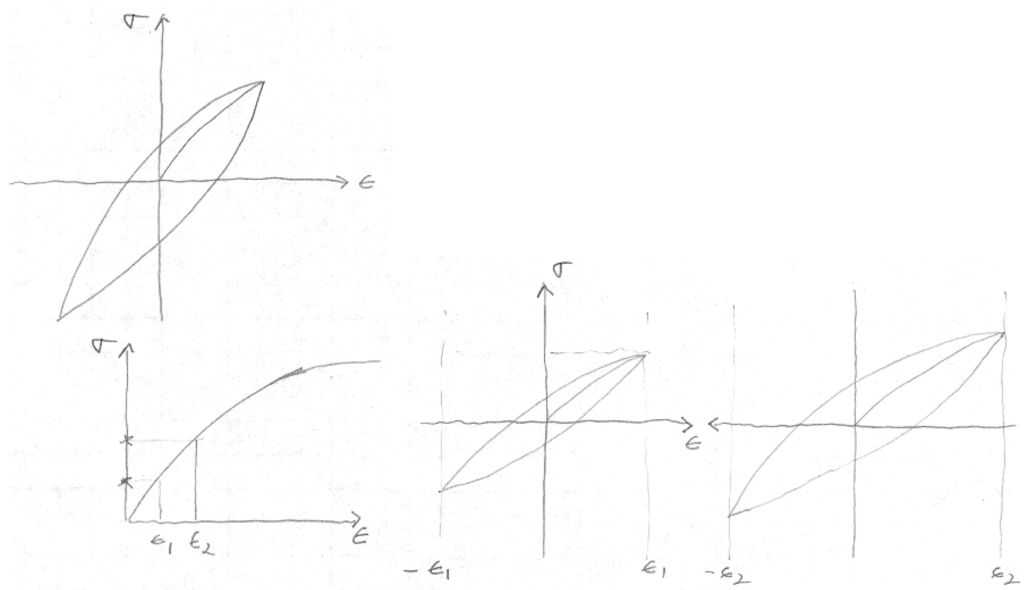


Fig. 7.7

Massing's hypothesis states that the stabilized hysterisis loop can be obtained by doubling the cyclic stress-strain curve as shown in Fig. 7.8.

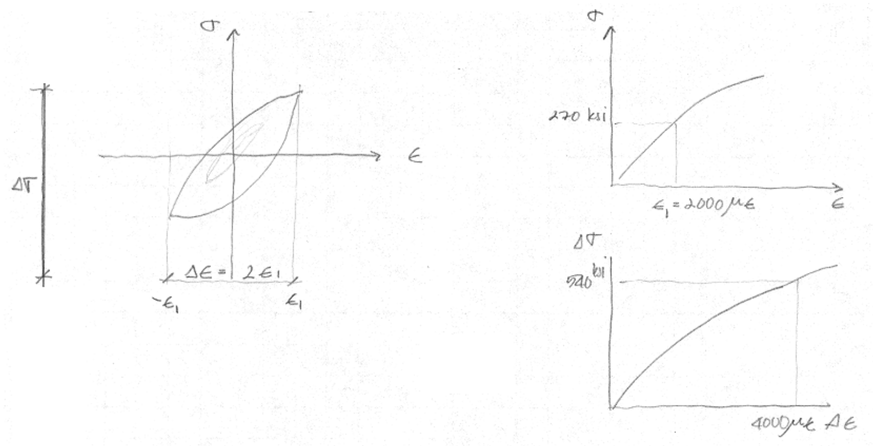


Fig. 7.8

b.) Incremental step method

A cyclic stress-strain curve is constructed by a sequence of gradually increasing or decreasing strain amplitude in a single test as shown by the strain-time curve in Fig. 7.9.

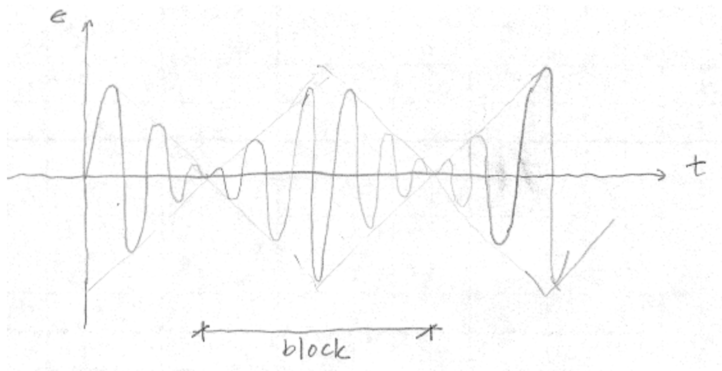


Fig. 7.9

In general, the material will be stabilized after three to four blocks of loading. After the incremental step test, the cyclic stress-strain curve will be nearly identical to the one obtained by connecting the loop tips.

7.5 $S-N$ Diagram and Stress Life Relation

The $S-N$ curve is a plot of alternating stress, S, versus cycles to failure, N_f obtained from the fatigue test. The $S-N$ data are usually presented on a log-log plot with the actual $S-N$ line representing the mean of the data as shown in Fig. 7.10.

Certain materials exhibit an endurance limit, which is stress level below that the material has an infinite life. For engineering purpose, the infinite life is usually considered as 10^6 cycles. For most nonferrous metal such as aluminum, there exist no distinct endurance limit and the $S-N$ curve has a continuous slope. A pseudo-endurance limit for these materials is taken on the stress value corresponding to $5(10)^8$ cycles.

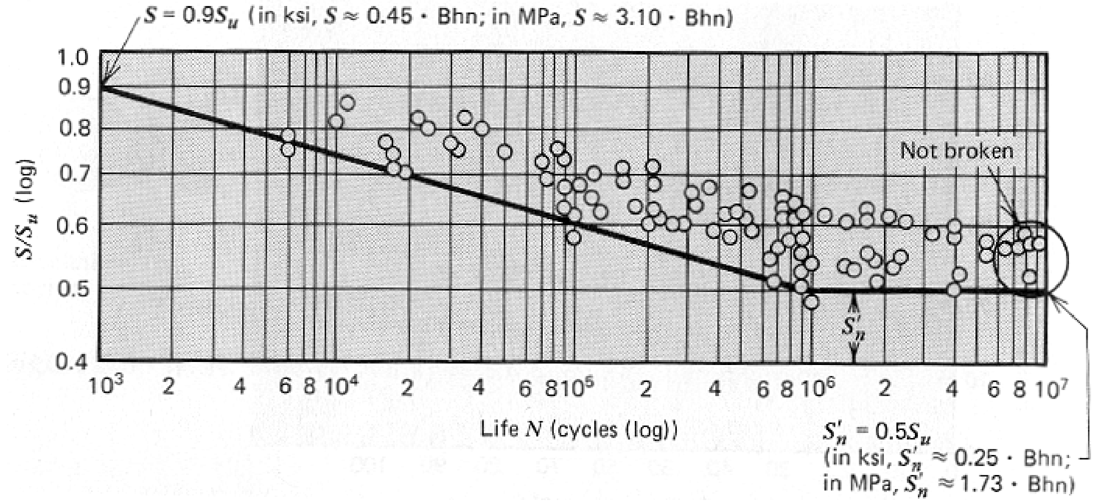


Fig. 7.10

Empirical relations of endurance limit are

$$S_e(\text{ksi}) = 0.25(BHN)$$

$$S_e(\text{MPa}) = 1.73(BHN)$$

where $BHN =$ brinell hardeness number. In addition,

$$S_e \approx 100\,\text{ksi} \qquad \text{for } BHN > 400$$

$$S_e \approx 0.5S_{ult} \qquad \text{for } S_{ult} < 200\,\text{ksi}$$

$$S_e \approx 100\,\text{ksi} \qquad \text{for } S_{ult} \geq 200\,\text{ksi}$$

Stress-Life relation

Consider a general $S-N$ curve plotted on a log-log coordinate as shown in Fig. 7.11.

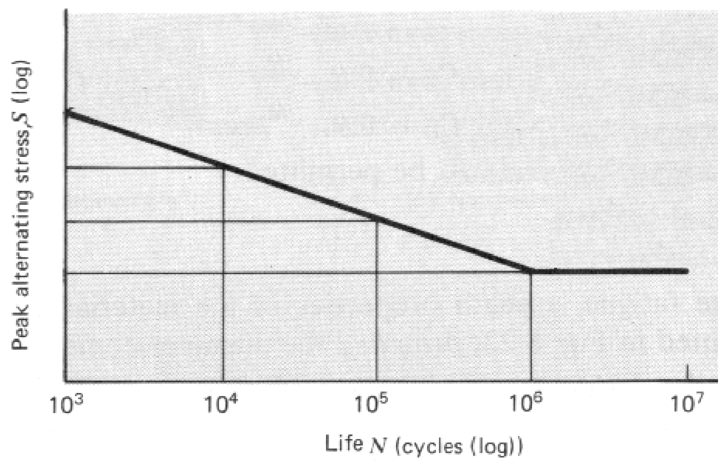


Fig. 7.11

For the number of cycle is $10^3 \leq N_f \leq 10^6$, the $S-N$ curve is a straight line. The following equation can be fitted to obtain a mathematical representation of the curve.

$$\begin{aligned}\log S &= b\log N_f + C\\ &= b\log N_f + \log 10^C\\ &= \log N_f^b + \log 10^C\\ &= \log N_f^b 10^C\end{aligned}$$

Thus, we have

$$S = 10^C N_f^b$$

The exponents b and C are determined by using two end points. In general, alternating stress level corresponding to a life of 10^3 cycles can be estimated as $S_{1000} = 0.9S_{ult}$. Also, the endurance limit at 10^6 cycles can be estimated as, $S_e = 0.5S_{ult}$. Thus,

$$\log S_e = b\log 10^6 + C = 6b + C$$

$$\log S_{1000} = b\log 10^3 + C = 3b + C$$

$$C = \log S_{1000} - 3b$$

$$\log S_e = 3b + \log S_{1000}$$

$$\log S_{1000} - \log S_e = -3b$$

$$b = -\frac{1}{3}\log\frac{S_{1000}}{S_e} = -0.085$$

$$C = \log\frac{S_{1000}^2}{S_e} = \log 1.62S_{ult}$$

Thus, the mathematical representation of the curve is

$$S = 1.62S_{ult}N_f^{-0.085}$$

The general form of the above equation may be written as

$$S = A(N_f)^B$$

In some cases, this equation is written in another form of

$$S = \sigma'_f(2N_f)^b$$

The fitting constant for of the two forms are related by

$$A = 2^b\sigma'_f \text{ and } B = b$$

and are given in Table 7-1 for several engineering metals.

It should be noted that the $S - N$ curve is primary valid within the elastic range of the material and does not work well in low-cycle fatigue.

Table 7-1

Material	Yield Strength σ_o	Ultimate Strength σ_u	True Fracture Strength $\tilde{\sigma}_{fB}$	$\sigma_a = \sigma'_f(2N_f)^b = AN_f^B$ σ'_f	A	$b = B$
(a) Steels						
AISI 1015 (normalized)	227 (33)	415 (60.2)	725 (105)	976 (142)	886 (128)	−0.14
Man-Ten (hot rolled)	322 (46.7)	557 (80.8)	990 (144)	1089 (158)	1006 (146)	−0.115
RQC-100 (roller Q & T)	683 (99.0)	758 (110)	1186 (172)	938 (136)	897 (131)	−0.0648
AISI 4142 (Q & T, 450 HB)	1584 (230)	1757 (255)	1998 (290)	1937 (281)	1837 (266)	−0.0762
AISI 4340 (aircraft quality)	1103 (160)	1172 (170)	1634 (237)	1758 (255)	1643 (238)	−0.0977
(b) Other Metals						
2024-T4 Al	303 (44.0)	476 (69.0)	631 (91.5)	900 (131)	839 (122)	−0.102
Ti-6Al-4V (solution treated and aged)	1185 (172)	1233 (179)	1717 (249)	2030 (295)	1889 (274)	−0.104

Example 7-1

From axially loaded fatigue testing under zero mean stress of unnotched AISI 4340 steel specimen, we obtain the stress amplitude and corresponding stress as shown in Table Ex 7-1. Plot the data on the log-log coordinate and determine the constant A and B of the equation $S = A(N_f)^B$.

Table Ex 7-1

σ_a (MPa)	N_f (Cycles)
948	222
834	992
703	6004
631	14130
579	43860
524	132150

The plotted data are shown in the Fig Ex 7-1a. They seem to fall along a straight line, and the first and the last points represent the line well. Using this two point and denoting them (σ_1, N_{f1}) and (σ_2, N_{f2}), we have

$$\sigma_1 = A(N_{f1})^B \qquad \sigma_2 = A(N_{f2})^B$$

It should be noted that for axially loaded fatigue testing, the nominal stress, S, is equal to the point stress, σ.

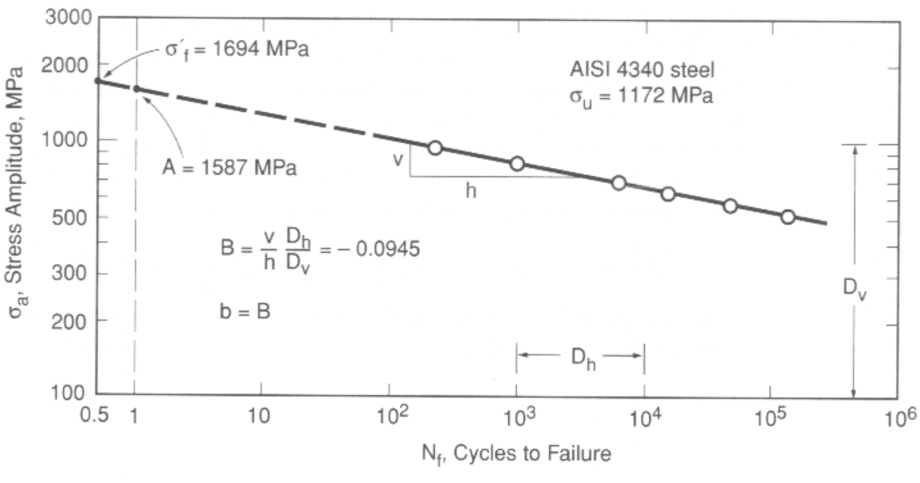


Fig Ex 7-1

Dividing the second equation into the first, and take logarithms of both sides.

$$\frac{\sigma_1}{\sigma_2} = \left(\frac{N_{f1}}{N_{f2}}\right)^B$$

$$\log\frac{\sigma_1}{\sigma_2} = B\log\frac{N_{f1}}{N_{f2}}$$

Solving for B,

$$B = \frac{\log\frac{\sigma_1}{\sigma_2}}{\log\frac{N_{f1}}{N_{f2}}} = \frac{\log\sigma_1 - \log\sigma_2}{\log N_{f1} - \log N_{f2}}$$

$$B = \frac{\log 948 - \log 524}{\log 222 - \log 132150} = -0.0928$$

Then, we can find A.

$$A = \frac{\sigma_1}{(N_{f1})^B} = \frac{948}{222^{-0.0928}} = 1565\ \text{MPa}$$

7.6 Fatigue Strength Diagram (Haigh diagram)

A fatigue strength diagram is a plot of alternating stress versus mean stress with lines of constant life.

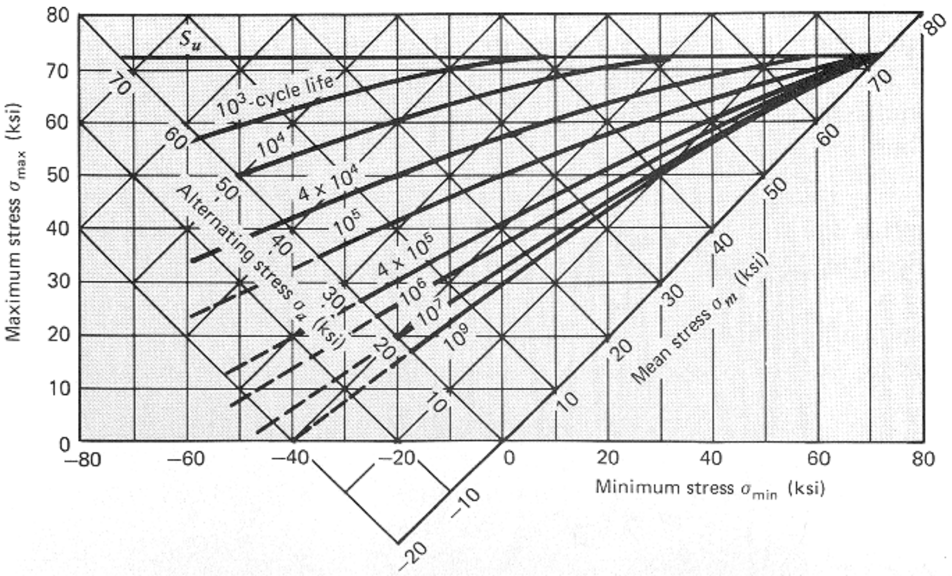


Fig. 7.12

Fig. 7.12 is a fatigue strength diagram for alloy steel, S_u = 125 to 180 ksi, axial loading. Since the tests required to generate a fatigue strength diagram is expensive, several relationships have been proposed to generate the lines defining the infinite life design region. The following relationships are commonly used for an infinite life.

a.) Soderberg (USA, 1930)

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = 1$$

b.) Gerber (German, 1874)

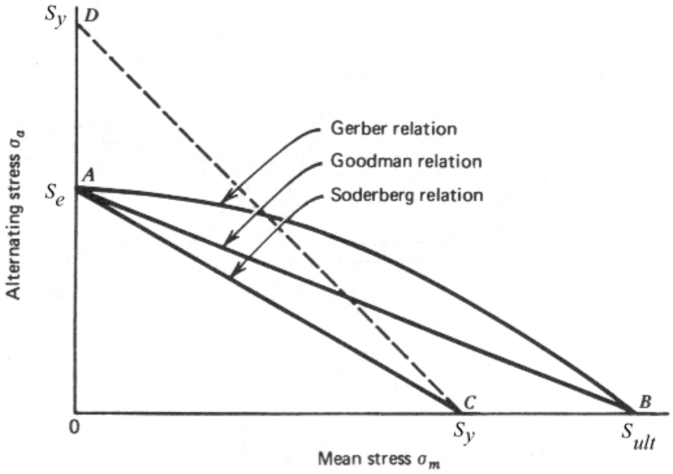
$$\frac{\sigma_a}{S_e} + \left(\frac{\sigma_m}{S_{ult}}\right)^2 = 1$$

c.) Goodman (England, 1899)

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ult}} = 1$$

The curves of each relationship are shown in Fig. 7.13. For finite life, the S_e in the above equations can be replaced with a fully reversed alternating stress level corresponding to that finite life. It should be noted that

1. The Soderberg model is too conservative and seldom used.
2. The Gerber model is good for ductile material.
3. The Goodman model is good for brittle material.



S_y
D
Alternating stress σ_a
Gerber relation
Goodman relation
Soderberg relation
A
S_e
0
C
S_y
B
S_{ult}
Mean stress σ_m

Fig. 7.13

Example 7-2

A 30 mm diameter shaft is subjected to the cyclic loading as shown in Fig. Ex 7-2a where the magnitude of the load varies from $P_{\min} = -0.60 P_{\max}$ to $P_{\max}$. The shaft is made of stress-relieved cold-worked SAE 1040 steel having the ultimate stress of 830 MPa, yielding stress of 660 MPa, the endurance limit of 410 MPa. Determine the magnitude of $P_{\max}$ based on a factor of safety of 1.80 against the failure at $N_f = 10^7$ cycles.

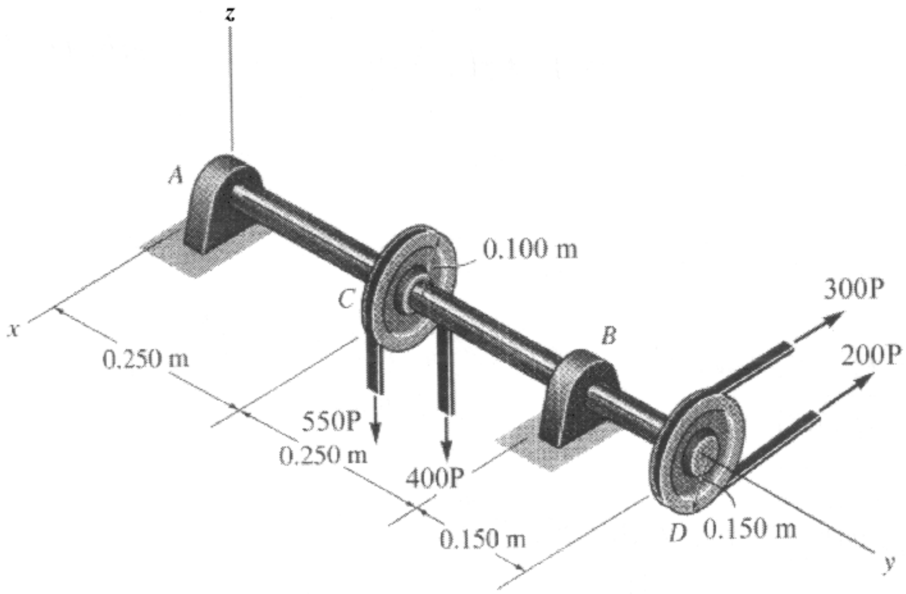


Fig. Ex 7-2a

Since the steel is a ductile material, we will use the Gerber relation to determine the magnitude of $P_{\max}$.

$$\frac{\sigma_a}{S_e} + \left(\frac{\sigma_m}{S_{ult}}\right)^2 = 1$$

For the linear elastic behavior of the material under the cyclic loading and we have the relationship of the minimum load and the maximum load in the form of $P_{\min} = -0.60 P_{\max}$, then, we obtain the relationship of the minimum stress and the maximum stress of the form

$$\sigma_{\min} = -0.60 \sigma_{\max}$$

The stress amplitude is $\sigma_a = \dfrac{\sigma_{\max} - \sigma_{\min}}{2}$. Then, we have

$$\sigma_{\max} = 1.25 \sigma_a$$

Since $\sigma_{\max} = \sigma_m + \sigma_a$, the mean stress is

$$\sigma_m = 0.25 \sigma_a$$

Thus, from the Gerber relation, we have

$$\frac{\sigma_a}{410}+\left(\frac{0.25\sigma_a}{830}\right)^2=1$$

Solving the polynomial equation, we obtain the alternating stress equal to $403.9\,\text{MPa}$ and the maximum stress and the maximum stress equal to

$$\sigma_{\max} == 1.25\sigma_a = 504.9\,\text{MPa}$$

$$\sigma_{\min} = -0.60\sigma_{\max} = -302.9\,\text{MPa}$$

Since $\sigma_{\max} < \sigma_y = 660\,\text{MPa}$, the failure would be by fatigue and not by the general yielding.

The loads need to be multiplied by 1.80 due to the factor of safety. Then, we can draw the bending moment diagram and torque diagram as shown in Fig. Ex 7-2b.

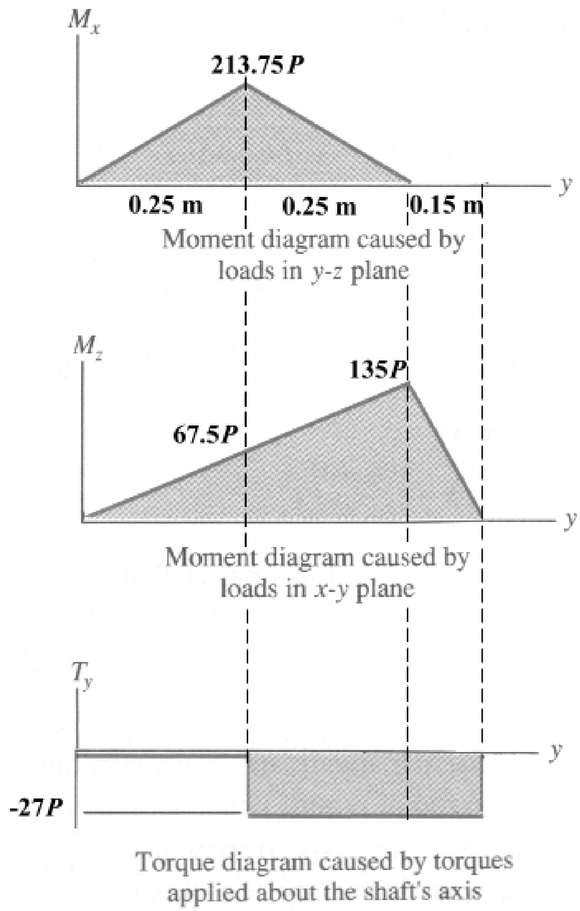


Fig. Ex 7-2b

The maximum bending moment occurred on the shaft is

$$M_{\max} = \sqrt{(213.75P)^2+(67.5P)^2} = 224.1P$$

The maximum toque occurred on the shaft is $27P$. The moment of inertia and the polar moment of inertia of the shaft are

$$I = \frac{\pi d^4}{64} = \frac{\pi(0.030)^4}{64} = 39.76(10^{-9})\,\text{m}^4$$

$$I = \frac{\pi d^4}{32} = \frac{\pi(0.030)^4}{32} = 79.52(10^{-9})\ \text{m}^4$$

Let $P = P_{\max}$. The flexural stress doe to the bending moment is

$$\sigma = \frac{Mc}{I} = \frac{224.15P_{\max}(0.015)}{39.76(10^{-9})} = 84.564(10^6)P_{\max}$$

and the shearing stress due to the torque is

$$\tau = \frac{Tc}{J} = \frac{27P_{\max}(0.015)}{79.52(10^{-9})} = 5.093(10^6)P_{\max}$$

For the steel, we use the maximum octahedral shearing stress criteria to predict $P_{\max}$.

$$3\tau^2 + \sigma^2 = \sigma_{\max}^2$$

$$P_{\max}^2 = 35.266$$

$$P_{\max} = 5.93\ \text{N}$$

$$P_{\min} = -3.56\ \text{N}$$

7.7 Endurance Limit Modifying Factor

The endurance limit of metallic materials is often obtained in the laboratory by a rotating beam specimen test. However, in most structural applications, the endurance limit of a structural member is obtained from modifying the data from the tests. There are many factors that affect the endurance limit. The followings are significant factors:

$$S_e = K_a K_b K_c K_d K_e S'_e$$

where $S_e =$ endurance limit of the structural member.

$S'_e =$ endurance limit of the test specimens.

$K_a =$ surface factor.

$K_b =$ size factor.

$K_c =$ load factor.

$K_d =$ temperature factor.

$K_e =$ other factor.

7.7.1 K_a, surface factor

$$K_a = aS_{ult}^b$$

Surface finish	*a*		*b*
	ksi	*MPa*	
Ground	1.34	1.58	-0.085
Machine or cold drawn	2.70	4.51	-0.265
Hot-rolled	14.4	57.7	-0.718
Forged	39.9	272	-0.995

7.7.2 K_b, size factor

For bending and torsion loading,

$$K_b = \begin{cases} \left[\dfrac{d}{0.3}\right]^{-0.1133} & 0.11 \le d \le 2\text{ in.} \\ \left[\dfrac{d}{7.62}\right]^{-0.1333} & 2.79 \le d \le 51\text{ mm} \end{cases}$$

For tension loading, $K_b = 1$.

7.7.3 K_c, load factor

$K_c = 0.923$ for axial loading when $S_{ult} \le 220$ ksi

$K_c = 1.0$ for axial loading when $S_{ult} > 220$ ksi

$K_c = 1.0$ for bending

$K_c = 0.577$ for torsion and shear

7.7.4 K_d, temperature factor

$$K_d = \frac{S_T}{S_{RT}}$$

where $S_T =$ tensile strength at the operating temperature

$S_{RT} =$ tensile strength at the room temperature

7.7.5 K_e, other factor

Only the stress concentration factor will be considered here.

$$K_e = \frac{1}{K_f}$$

where $K_f =$ fatigue stress concentration factor,

$$K_f = 1 + q(K_t - 1)$$

$q =$ material parameter

$K_t =$ static stress concentration factor

7.8 Fatigue Crack Propagation

The presence of a crack can significantly reduce the strength of an engineering component due to brittle fracture. However, it is unusual for a crack of dangerous size to exist initially. Normally, the crack is developed from a small flaw until it reaches the critical size. Crack propagation can be caused by cyclic loading. Typical constant amplitude crack propagation is shown in Fig. 7.14.

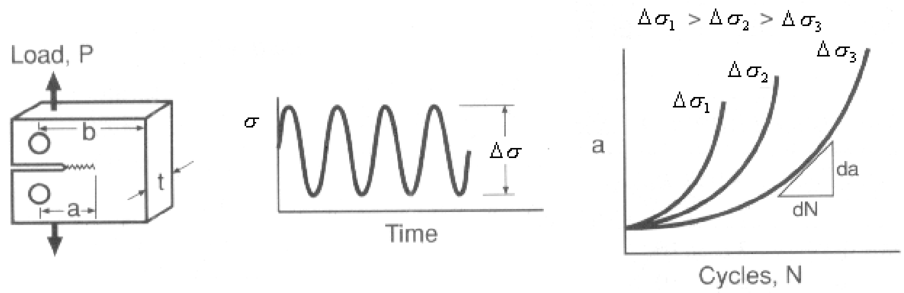


Fig. 7.14

Crack propagation rate is defined as crack extension per cycle, $\frac{da}{dN}$. The growth rate of crack is a function of stress intensity factor.

$$\frac{da}{dN} = f(K)$$

Fig. 7.15 is the typical experimental data plot of the crack propagation rate $\frac{da}{dN}$ versus stress intensity range ΔK .

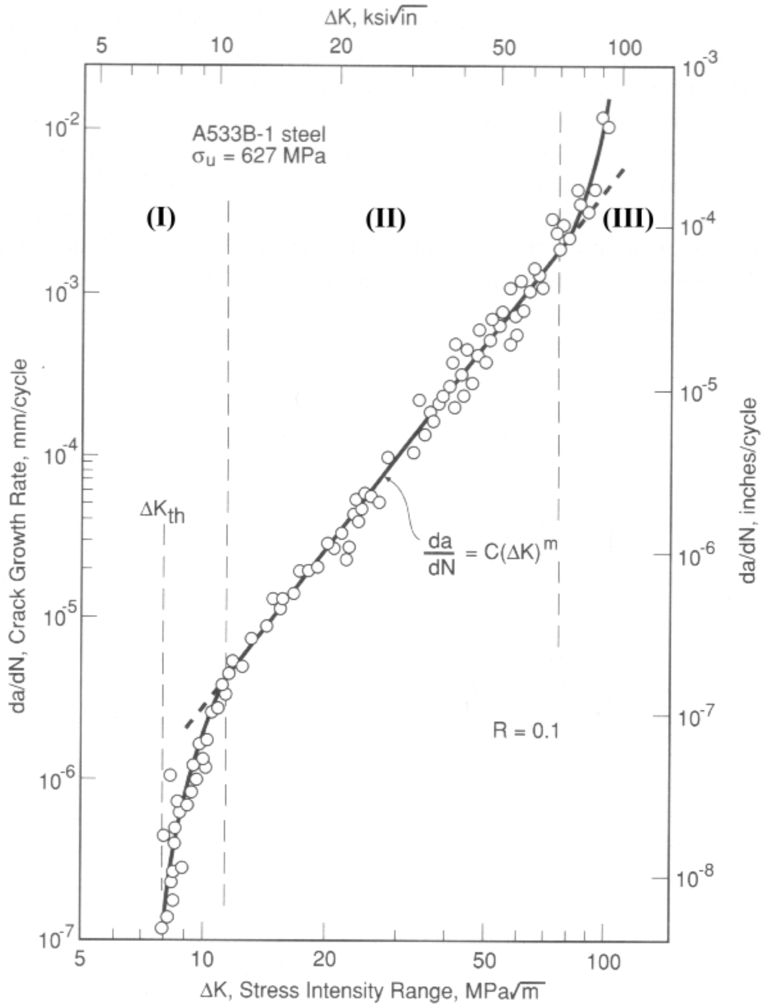


Fig. 7.15

It can be seen that the plot can be separated into three regions.

Region I: Crack behavior is associated with fatigue crack growth threshold value ΔK_{th} below which the crack growth is negligible.

$$\frac{da}{dN} < 10^{-8} \text{ in/cycle}$$

Region II: The relationship between $\log \frac{da}{dN}$ versus $\log \Delta K$ is linear and steeper than the curve in Region I. This is due to rapid unstable crack growth just prior to final failure of the test specimen

$$10^{-4} < \frac{da}{dN} < 10^{-7} \text{ in/cycle}$$

The relationship representing this line is

$$\frac{da}{dN} = C(\Delta K)^m$$

where C is a constant and m is the slope on the log-log plot, assuming that the decades on both log scales are the same length. The value of m is important since it indicates the degree of sensitivity of the growth rate of the stress. For example, if $m = 3$, doubling the stress range ΔS doubles the stress intensity range, thus increasing the growth rate by a factor 0f $2^m = 8$.

Region III: The crack growth rate is extremely high and little fatigue life is involved.

7.9 Factors Affecting the Fatigue Crack Growth

Stress ratio effect

$$R = \frac{\sigma_{\min}}{\sigma_{\max}}$$

For a constant K, the more positive R, the higher crack growth rate as shown in Fig. 7.16.

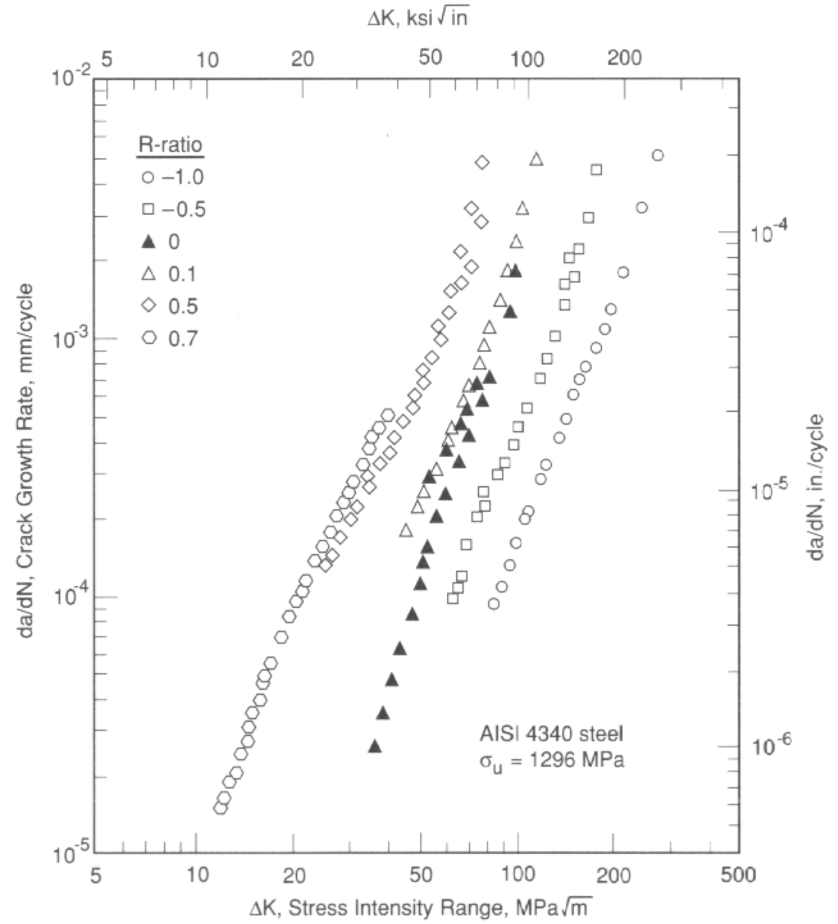


Fig. 7.16

Two crack growth models accounting for the stress ratio effect are

1. Forman's crack growth model

$$\frac{da}{dN} = C\frac{\Delta K^m}{(1-R)K_c - \Delta K}$$

where K_c is the fracture toughness for the plane stress condition.

2. Walker's crack growth model

$$\frac{da}{dN} = C\left[(1-R)^m K_{\max}\right]^n$$

Frequency effect

At normal environmental condition, frequency has little effect on fatigue life for metallic structure. However, the growth rate will be significantly affected if under an adverse environment.

Temperature effect

Fatigue life will be reduced if the temperature is increased as shown in Fig. 7.17.

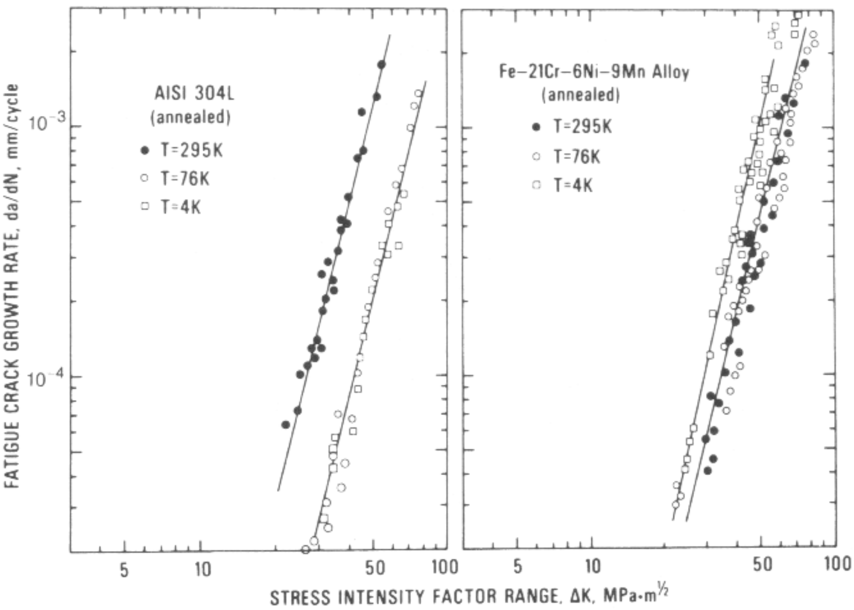


Fig. 7.17

Chapter 8

Beams on Elastic Foundation

8.1 Introduction

In some applications such as railroad track and the rail of movable crane, the rail does usually acts as a beam of relatively small flexural stiffness placed on elastic foundation. The loads are transferred through the beam to the elastic foundation. This rail can be analyzed as a beam supported by series of discrete elastic springs. However, this analytical method is very tedious. It is usually more practical to idealize the supports as a continuous elastic foundation.

8.2 General Theory

The response of a beam on elastic foundation can be depicted by a single differential equation subject to different boundary conditions.

Assumptions

1. The foundation has sufficient strength to prevent failure.
2. The foundation behaves linearly elastic under loads with a small deflection.
3. The beam is fully attached to the foundation.

Consider a beam of infinite length resting on an elastic foundation with infinite length and subjected to a point load P acting at the origin of the coordinate (x, y, z) as shown in Fig. 8.1a. Under the action of the load P, the beam is deflected as shown in Fig. 8.1c, which induces a distributed force q between the beam and the foundation.

Consider a free body diagram of an element Δz as shown in Fig. 8.1b subjected to the positive shear forces and moments. For small displacement analysis, we have the differential relation of the displacement and the forces as

$$\frac{dy}{dz} = \theta$$

$$EI_x \frac{d^2 y}{dz^2} = -M_x \qquad 8.1$$

$$EI_x \frac{d^3 y}{dz^3} = -V_y$$

$$EI_x \frac{d^4 y}{dz^4} = -q$$

where the distributed reaction force q is positive when acting upward.

For linearly elastic foundation, the distributed force q is linearly proportional to the deflection y. Thus,

$$q = ky \qquad 8.2$$

$$k = bk_o \tag{8.3}$$

where k is the elastic coefficient, k_o is the elastic foundation modulus, and b is the width of the foundation.

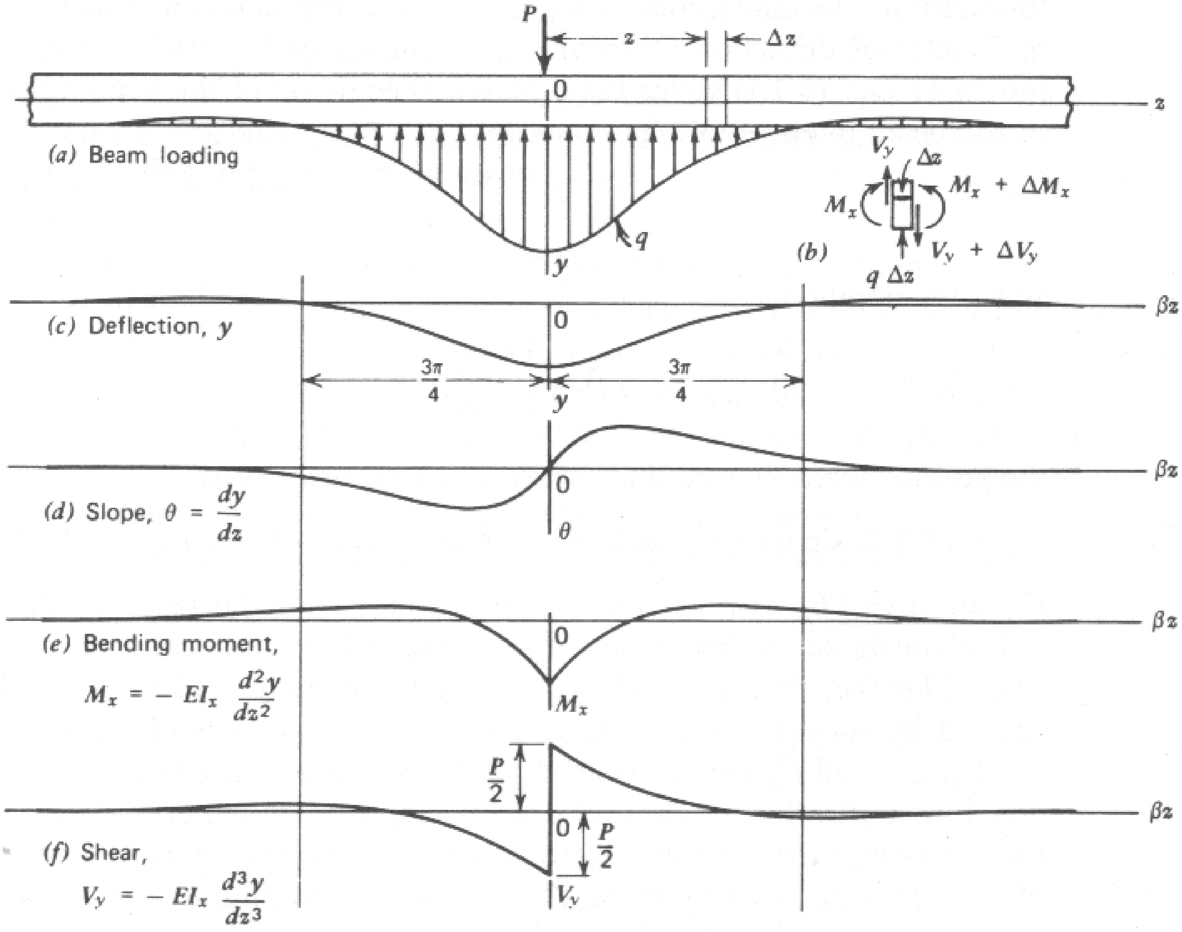


Fig. 8.1

The k_o usually has a value between $20(10^6)\text{N/m}^2\text{/m}$ to $200(10^6)\text{N/m}^2\text{/m}$ for soil. Large values of k are best. Then,

$$EI_x \frac{d^4y}{dz^4} = -ky \tag{8.4}$$

$$\frac{d^4y}{dz^4} = -\frac{k}{EI_x}y$$

To solve this homogeneous, fourth order, linear differential equation with constant coefficients, we let $\dfrac{k}{EI_x} = 4\beta^4$, then,

$$\frac{d^4y}{dz^4} + 4\beta^4 y = 0 \tag{8.5}$$

By using the method of differential equations or by direct substituting into Eq. 8.5, the general solution of y is

$$y = e^{\beta z}(C_1 \sin\beta z + C_2 \cos\beta z) + e^{-\beta z}(C_3 \sin\beta z + C_4 \cos\beta z) \qquad 8.6$$

where the constant of integration C_1, C_2, C_3, and C_4 can be determined by using the boundary conditions of the beam.

Consider a semi-infinite beam on elastic foundation as shown in Fig. 8.2, which is a half of the beam in Fig. 8.1. Since the deflection $y = 0$ when $z \to \alpha$, then, the term $e^{\beta z} \to \alpha$ and $e^{-\beta z} \to 0$. Hence, we obtain $C_1 = C_2 = 0$ and

$$y = e^{-\beta z}(C_3 \sin\beta z + C_4 \cos\beta z) \qquad z \geq 0 \qquad 8.7$$

Due to the symmetry of the beam, we can determine the deflection of the beam for negative value of z by $y(-z) = y(z)$.

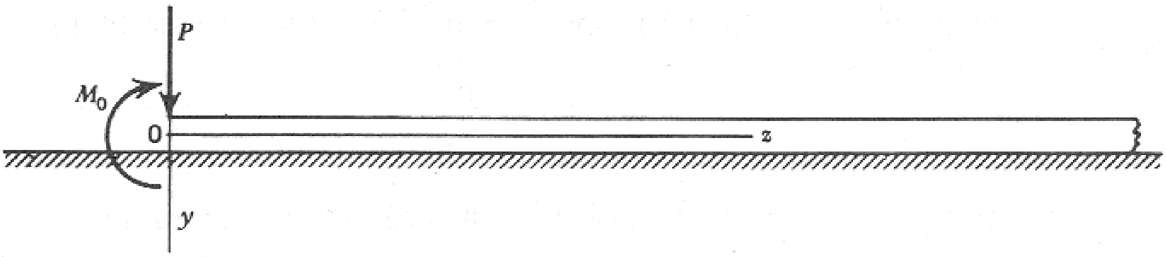


Fig. 8.2

8.3 Infinite Beam Subjected to Point Load

The constant of integration C_3 and C_4 of the Eq. 8.7 can be determined by using the following boundary conditions of the infinite beam:

1. The slope of the beam $\dfrac{dy}{dz} = 0$ at $z = 0$ due to the symmetry of the beam.

$$\frac{dy}{dz} = -\beta e^{-\beta z}(C_3 \sin\beta z + C_4 \cos\beta z) + \beta e^{-\beta z}(C_3 \cos\beta z - C_4 \sin\beta z) = 0$$

$$C_3 = C_4 = C$$

$$y = Ce^{-\beta z}(\sin\beta z + \cos\beta z) \qquad 8.8$$

2. A half of the point load P is carried by the beam specified by $+z$ and the other half is carried by the beam specified by $-z$.

$$2\int_0^{\alpha} ky\,dz = P$$

$$2\int_0^{\alpha} kCe^{-\beta z}(\sin\beta z + \cos\beta z)dz = P$$

$$\int_0^{\alpha} e^{-\beta z}(\sin\beta z)dz + \int_0^{\alpha} e^{-\beta z}(\cos\beta z)dz = \frac{P}{2kC}$$

$$\frac{1}{2\beta}+\frac{1}{2\beta}=\frac{P}{2kC}$$

$$C=\frac{P\beta}{2k} \qquad 8.9$$

Therefore, the deflection, the slope, the moment, and the shear of the beam can be written as

$$y=\frac{P\beta}{2k}e^{-\beta z}(\sin\beta z+\cos\beta z) \qquad z\geq 0 \qquad 8.10$$

$$\begin{aligned}\theta&=\frac{dy}{dz}=\frac{P\beta}{2k}[-2\beta e^{-\beta z}(\sin\beta z)]\\&=-\frac{P\beta^2}{k}[e^{-\beta z}\sin\beta z]\end{aligned} \qquad z\geq 0 \qquad 8.11$$

$$\begin{aligned}M_x&=-EI_x\frac{d^2y}{dz^2}=EI_x\frac{P\beta^2}{k}\frac{d}{dz}[e^{-\beta z}(\sin\beta z)]\\&=EI_x\frac{P\beta^2}{k}[\beta e^{-\beta z}(\cos\beta z-\sin\beta z)]\\&=EI_x\frac{P\beta^3}{k}[e^{-\beta z}(\cos\beta z-\sin\beta z)]\\&=\frac{P}{4\beta}[e^{-\beta z}(\cos\beta z-\sin\beta z)]\end{aligned} \qquad z\geq 0$$

8.12

$$\begin{aligned}V_y&=-\frac{dM}{dz}=-\frac{d}{dz}\left[\frac{P}{4\beta}[e^{-\beta z}(\cos\beta z-\sin\beta z)]\right]\\&=-\frac{P}{4\beta}[\beta e^{-\beta z}(2\cos\beta z)]\\&=-\frac{P}{2}[e^{-\beta z}\cos\beta z]\end{aligned} \qquad z\geq 0 \qquad 8.13$$

Defining

$$A_{\beta z}=e^{-\beta z}(\sin\beta z+\cos\beta z) \qquad B_{\beta z}=e^{-\beta z}\sin\beta z$$

$$C_{\beta z}=e^{-\beta z}(\cos\beta z-\sin\beta z) \qquad D_{\beta z}=e^{-\beta z}\cos\beta z \qquad 8.14$$

Then, we have

$$y=\frac{P\beta}{2k}A_{\beta z} \qquad z\geq 0 \qquad 8.10$$

$$\theta=-\frac{P\beta^2}{k}B_{\beta z} \qquad z\geq 0 \qquad 8.11$$

$$M_x=\frac{P}{4\beta}C_{\beta z} \qquad z\geq 0 \qquad 8.12$$

$$V_y=-\frac{P}{2}D_{\beta z} \qquad z\geq 0 \qquad 8.13$$

Example 8-1

A rail road uses steel rails ($E = 200\,\text{GPa}$) with a depth of $184\,\text{mm}$. The distance from the top of the rail to its centroid is $99.1\,\text{mm}$, and the moment of inertia of the rail is $36.(10^6)\,\text{mm}^4$. The rail is supported by ties, ballast, and a road bed that together are assumed to act as an elastic foundation with spring constant $k = 14.0\,\text{N/mm}^2$.

a.) Determine the maximum deflection, maximum bending moment, and maximum flexural stress in the rail for a single wheel load of $170\,\text{kN}$ as shown in Fig. Ex 8-1a.

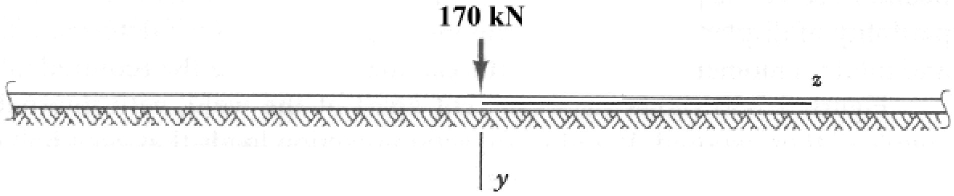


Fig. Ex 8-1a

b.) If a locomotive has 3 wheels per truck equally spaced at $1.70\,\text{m}$, determine the maximum deflection, maximum bending moment, and maximum flexural stress in the rail when the load on each wheel is $170\,\text{kN}$.

Since the equations of deflection and bending moment require the value of β,

$$\beta = \sqrt[4]{\frac{k}{4EI_x}} = \sqrt[4]{\frac{14}{4(200)10^3(36.9)10^6}} = 0.000830\,\text{mm}^{-1}$$

a.) The maximum deflection and the maximum bending moment occur under the load where

$$A_{\beta z} = C_{\beta z} = 1.0$$

Thus,

$$y_{\max} = \frac{P\beta}{2k}A_{\beta z} = \frac{170(10^3)0.000830}{2(14)}(1) = 5.039\,\text{mm}$$

$$M_{\max} = \frac{P}{4\beta}C_{\beta z} = \frac{170(10^3)}{4(0.000830)}(1) = 51.21\,\text{kN-m}$$

$$\sigma_{\max} = \frac{M_{\max}c}{I_x} = \frac{51.21(10^6)99.1}{36.9(10^6)} = 137.5\,\text{MPa}$$

b.) The deflection and the bending moment at any section of the beam obtained by superposition the effects of each wheel of the 3 wheel loads.

By using the superposition method, the maximum deflection and the maximum bending moment may be under 2 following cases:

1. Under one of the end wheel as shown in Fig. Ex 8-1b.

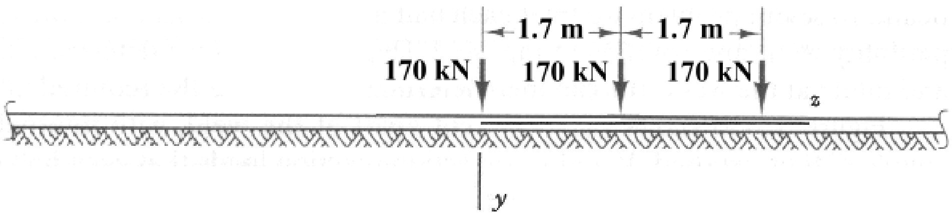


Fig. Ex 8-1b

2. Under the center wheel as shown in Fig. Ex 8-1c

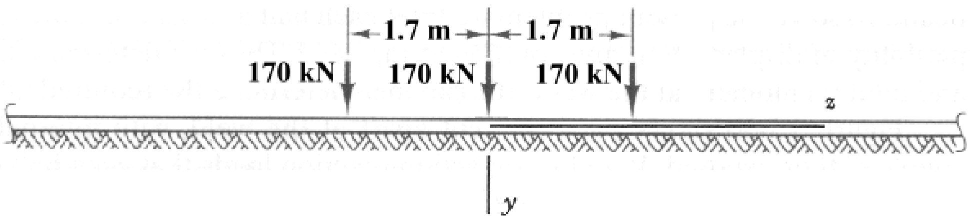


Fig. Ex 8-1c

For case 1, let the origin of the coordinate be located under one of the end wheel. The distance to the first wheel $z_1 = 0$, we have

$$A_{\beta z1} = C_{\beta z1} = 1.0$$

The distance from the origin to the next wheel is $z_2 = 1700\,\text{mm}$, we have

$$A_{\beta z2} = 0.2797 \qquad C_{\beta z2} = -0.2018$$

The distance from the origin to the next wheel is $z_3 = 3400\,\text{mm}$, we have

$$A_{\beta z3} = -0.0377 \qquad C_{\beta z3} = -0.0752$$

Therefore, for this case, we get the maximum deflection and the maximum bending moment equal to

$$y_{\max} = \frac{P\beta}{2k}(A_{\beta z1} + A_{\beta z2} + A_{\beta z3}) = 5.039(1 + 0.2797 - 0.0377) = 6.258\,\text{mm}$$

$$M_{\max} = \frac{P}{4\beta}(C_{\beta z1} + C_{\beta z2} + C_{\beta z3}) = 51.20(10^6)(1 - 0.2018 - 0.0752) = 37.02\,\text{kN - m}$$

For case 2, let the origin of the coordinate be located under the center wheel. The distance to the first wheel $z_1 = 0$, we have

$$A_{\beta z1} = C_{\beta z1} = 1.0$$

The distance from the origin to either of the end wheel is $z_2 = 1700\,\text{mm}$, we have

$$A_{\beta z2} = 0.2797 \qquad C_{\beta z2} = -0.2018$$

Therefore, for this case, we get the maximum deflection and the maximum bending moment equal to

$$y_{\max} = \frac{P\beta}{2k}(A_{\beta z1} + 2A_{\beta z2}) = 5.039(1 + 2(0.2797)) = 7.858\,\text{mm}$$

$$M_{\max} = \frac{P}{4\beta}(C_{\beta z1} + 2C_{\beta z2}) = 51.20(10^6)(1-2(0.2018)) = 30.54 \text{ kN - m}$$

From the calculation, we obtain

$$y_{\max} = 7.858 \text{ mm}$$

$$M_{\max} = 37.02 \text{ kN - m}$$

and
$$\sigma_{\max} = \frac{M_{\max} c}{I_x} = \frac{37.02(10^6)99.1}{36.9(10^6)} = 99.4 \text{ MPa}$$

8.4 Beam Supported on Equally Spaced Separated Elastic Supports

The concept of beam on elastic foundation previously mentioned can be applied to the problem of long beam supported by elastic supports equally spaced along the beam as the one shown in Fig. 8.3.

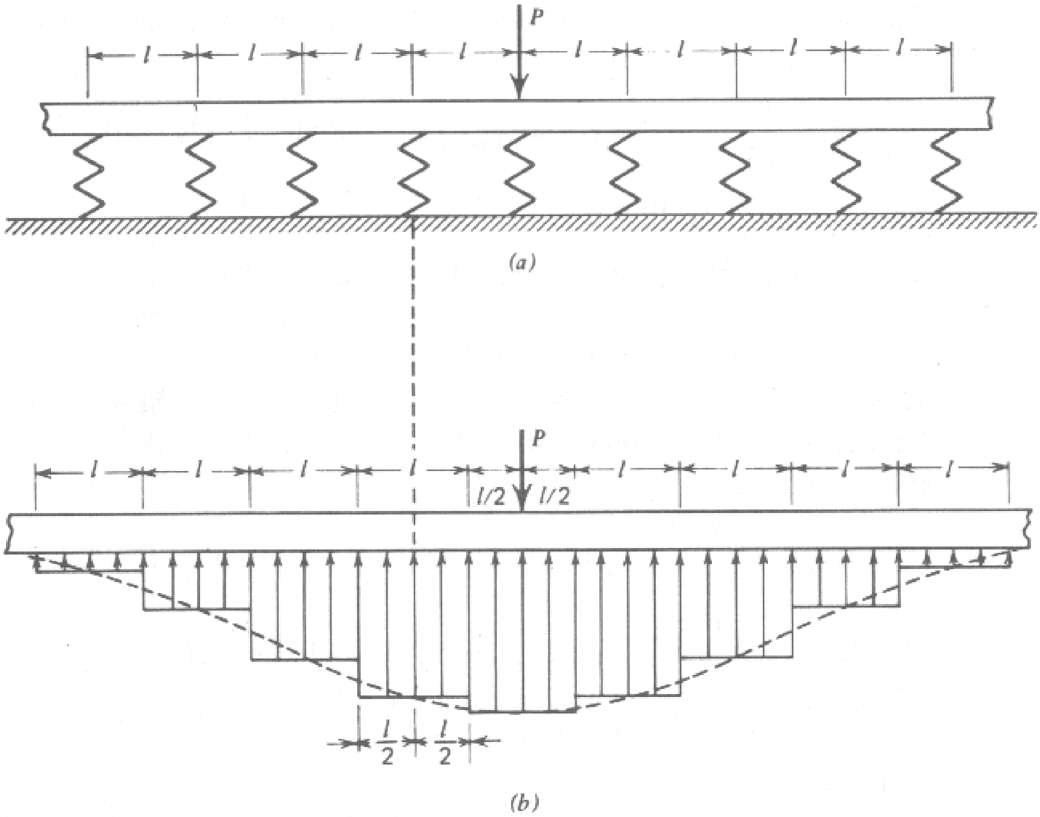


Fig. 8.3

Let each spring in Fig. 8.3a has the same spring constant K. The reaction force R that each spring exert on the beam is directly proportional to the deflection y of the beam at the section where the spring is attached. Thus,

$$R = Ky$$

If l is the spring spacing, the load R can be idealized as uniformly distributed over a total span l ($l/2$ on either side of the spring) as shown in Fig. 8.3b. If the stepped distributed loading is approximated by the dashed curve, the approximate distributed load is similar to the distributed load q of Fig. 8.1a. If the two forces are to be the same, then,

$$Ky = kyl$$

$$k = \frac{K}{l}$$

where k is the elastic coefficient for this case. Then, we can use Eq. 8-10 to Eq. 8-13 to find the deflection, the slope, the moment, and the shear of the beam. However, it has been found that the solutions are only practically useful when

$$l \le \frac{\pi}{4\beta}$$

The approximate solution for a beam of infinite length, with equally spaced elastic supports, may be used to obtain a reasonable approximate solution for a sufficiently long finite length beam as shown in Fig. 8.4a. In general, the end springs do not coincide with the ends of the beam, but lie at some distance less than $l/2$ from the end of the beam as shown in Fig. 8.4b. Thus, we extend the beam of length L to a beam of length L'', where

$$L'' = ml$$

and integer m is the number of spring supports. To obtain a reasonable approximate solution,

$$L'' \ge \frac{3\pi}{2\beta}$$

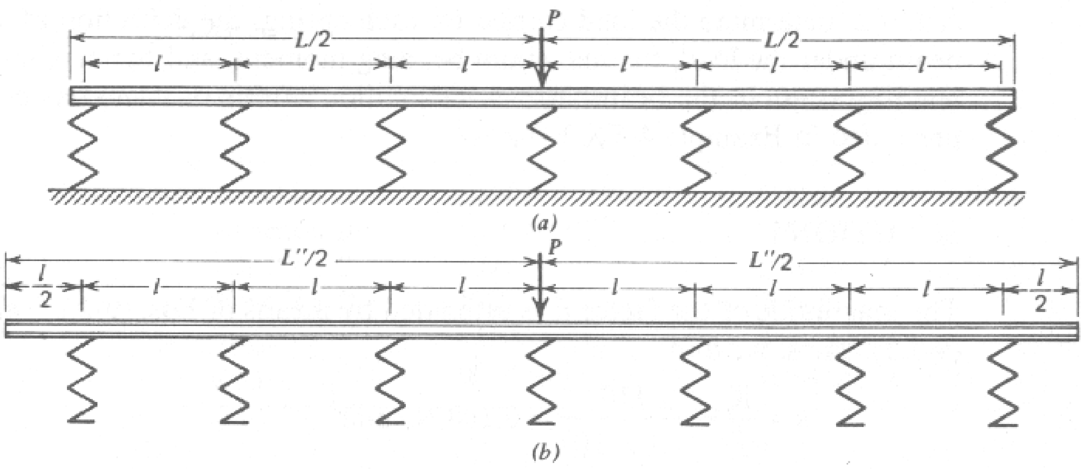


Fig. 8.4

Example 8-2

An aluminum alloy I-beam (depth = $100\,\text{mm}$, $I_x = 2.45(10^6)\,\text{mm}^4$, $E = 72\,\text{GPa}$) as shown in Fig. Ex 8-2 has a length of $L = 6.6\,\text{m}$ and is supported by 7 springs ($K = 110\,\text{N/mm}$) spaced at distance $l = 1.10\,\text{m}$ center to center along the beam. A load $P = 12.0\,\text{kN}$ is applied at the center of the beam. Determine the load carried by each spring, the maximum deflection of the beam, the maximum bending moment, and the maximum bending stress in the beam.

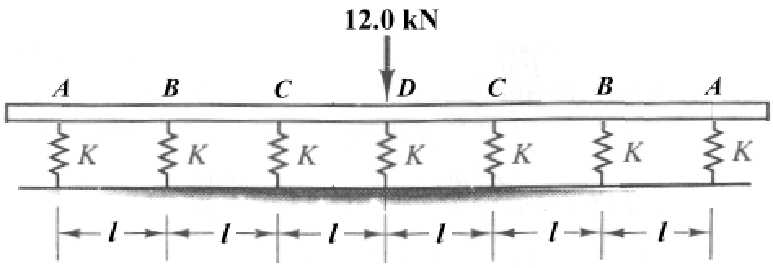


Fig. Ex 8-2

The elastic coefficient,

$$k = \frac{110}{1100} = 0.100\,\text{N/mm}^2$$

and the value of β,

$$\beta = \sqrt[4]{\frac{0.100}{4(72)10^3(2.45)10^6}} = 0.000614\,\text{mm}^{-1}$$

Check the spacing of the spring.

$$l < \frac{\pi}{4\beta} = \frac{\pi}{4(0.000614)} = 1279\,\text{mm} \qquad \text{O.K.}$$

Check the length of the beam.

$$L'' = 6600 + 1100 = 7700\,\text{mm} > \frac{3\pi}{2\beta} = \frac{3\pi}{2(0.000614)} = 7675\,\text{mm} \qquad \text{O.K.}$$

The maximum deflection and the maximum bending moment of the beam occur under the load where

$$A_{\beta z} = C_{\beta z} = 1.0$$

Thus,

$$y_{\max} = \frac{P\beta}{2k}A_{\beta z} = \frac{12(10^3)0.000614}{2(0.10)}(1) = 36.84\,\text{mm}$$

$$M_{\max} = \frac{P}{4\beta}C_{\beta z} = \frac{12(10^3)}{4(0.000614)}(1) = 4.886(10^6)\,\text{N - m}$$

$$\sigma_{\max} = \frac{M_{\max} c}{I_x} = 99.7\ \text{MPa}$$

Due to the symmetry of the beam, the magnitude of βz, the corresponding $A_{\beta z}$, and the deflection for the first (point C), second (point B), and third (point A) springs to the right and left of the load are

$$\beta l = 0.6754 \quad A_{\beta l} = 0.7153 \quad y_C = \frac{P\beta}{2k} A_{\beta l} = 36.84(0.7153) = 26.53\ \text{mm}$$

$$2\beta l = 1.3508 \quad A_{2\beta l} = 0.3094 \quad y_B = \frac{P\beta}{2k} A_{2\beta l} = 36.84(0.3094) = 11.40\ \text{mm}$$

$$3\beta l = 2.0262 \quad A_{3\beta l} = 0.0605 \quad y_A = \frac{P\beta}{2k} A_{3\beta l} = 36.84(0.0605) = 2.23\ \text{mm}$$

The reaction for each spring can be obtained by using the equation $R = Ky$ and the results are shown in the Table Ex 8-2.

Table Ex 8-2

	Approximate solution	Exact solution
Reaction A	245 N	-454 N
Reaction B	1254 N	1216 N
Reaction C	2899 N	3094 N
Reaction D	4052 N	4288 N
$y_{\max}$	36.84 mm	38.98 mm
$M_{\max}$	4.886 N - m	4.580 N - m

Comparing the results with the exact results by using the energy method, we can see that only the reaction at A are considerably in error.

8.5 Infinite Beam Subjected to a Distributed Load Segment

Consider the beam on elastic foundation subjected to uniformly distributed load segment as shown in Fig. 8.5.

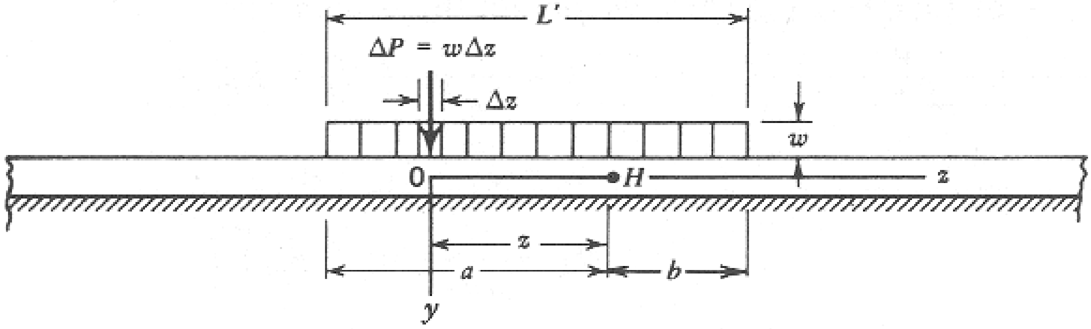


Fig. 8.5

From the displacement solution of the beam subjected to concentrated load, Eq. 8.10,

$$y = \frac{P\beta}{2k} e^{-\beta z} (\sin \beta z + \cos \beta z) \qquad z \geq 0 \qquad 8.10$$

Then,

$$dy_H = \frac{\beta w}{2k} e^{-\beta z} (\sin \beta z + \cos \beta z) dz$$

By using the principle of superposition, the total deflection due to the distributed load is

$$y_H = \int_0^a \frac{\beta w}{2k} e^{-\beta z} (\sin \beta z + \cos \beta z) dz$$

$$+ \int_0^b \frac{\beta w}{2k} e^{-\beta z} (\sin \beta z + \cos \beta z) dz$$

$$y_H = \frac{w\beta}{2k} \left[\frac{1}{\beta} \left(1 - e^{-a\beta} \cos \beta a\right) + \frac{1}{\beta} \left(1 - e^{-b\beta} \cos \beta b\right) \right]$$

$$= \frac{w}{2k} \left[2 - e^{-\beta a} \cos \beta a - e^{-\beta b} \cos \beta b\right] \qquad 8.11$$

Then, by using the differential relations, the deflection, the slope, the shear force, and the bending moment of the beam can be determined and simplified as

$$y_H = \frac{w}{2k} \left[2 - D_{\beta a} - D_{\beta b}\right]$$

$$\theta_H = \int_0^a \frac{dy_H}{dz} dz + \int_0^b \frac{dy_H}{dz} dz = \frac{w\beta}{2k} \left[A_{\beta a} - A_{\beta b}\right] \qquad 8.12$$

$$M_H = \frac{w}{4\beta^2} \left[B_{\beta a} + B_{\beta b}\right]$$

$$V_H = \frac{w}{4\beta} \left[C_{\beta a} - C_{\beta b}\right]$$

where

$$A_{\beta a} = e^{-\beta a}(\sin\beta a + \cos\beta a) \qquad B_{\beta a} = e^{-\beta a}\sin\beta a$$

$$C_{\beta a} = e^{-\beta a}(\cos\beta a - \sin\beta a) \qquad D_{\beta a} = e^{-\beta a}\cos\beta a$$

$$A_{\beta b} = e^{-\beta b}(\sin\beta b + \cos\beta b) \qquad B_{\beta b} = e^{-\beta b}\sin\beta b$$

$$C_{\beta b} = e^{-\beta b}(\cos\beta b - \sin\beta b) \qquad D_{\beta b} = e^{-\beta b}\cos\beta b$$

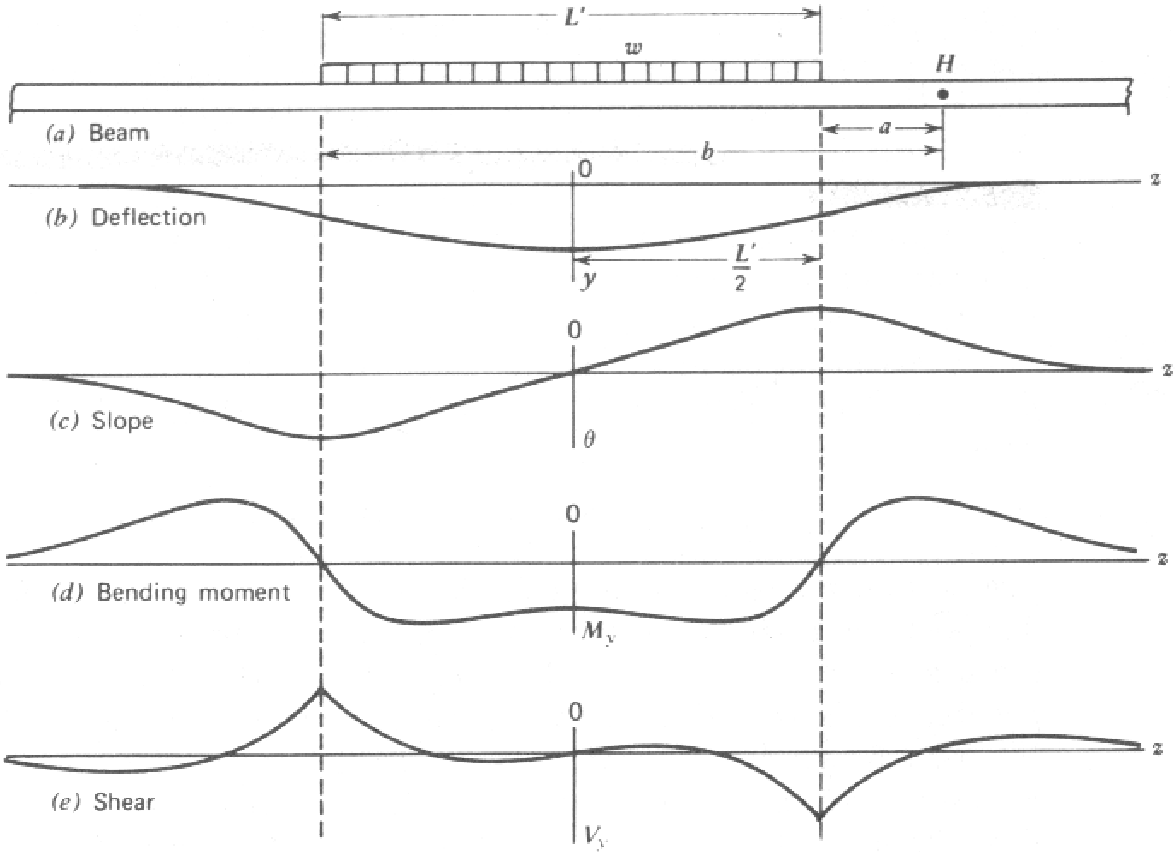


Fig. 8.6

Fig. 8.6 shows the plot of the expressions of the deflection, the slope, the shear force, and the bending moment of the beam with respect to the z axis. We can see that the maximum deflection occurs at the center of the segment L'. However, the location of the maximum bending moment may or may not occur at the center of the segment L', depending on the magnitude of $\beta L'$.

If $\beta L' \leq \pi$, then, the location of the maximum bending moment is at the center of the segment L'.

If $\beta L' \rightarrow \alpha$, then, $\theta \rightarrow 0$, $M_x \rightarrow 0$, $V_y \rightarrow 0$, and $y \rightarrow \dfrac{w}{k}$. Therefore, the location of the maximum bending moment of the beam is at either $\beta a = \pi/4$ or $\beta b = \pi/4$.

If $\beta L' > \pi$, then, the location of the maximum bending moment may lie outside the segment L'. However, the maximum value outside the segment L' is larger than the maximum value within the segment L' only 3%. Thus, we may assume that the location of the maximum bending moment in this case is at $\pi / 4\beta$ from either ends of the uniformly distributed load within the segment L'.

Example 8-3

A long wood beam ($E = 10.0\,\text{GPa}$) has a rectangular cross section with a depth of $200\,\text{mm}$ and a width of $100\,\text{mm}$. It rests on an earth foundation having spring constant of $k_o = 0.040\,\text{N/mm}^3$ and is subjected to a uniformly distributed load $w = 35.0\,\text{N/mm}$ extending over a length $L' = 3.61\,\text{m}$. Taking the origin of the coordinate at the center of the segment L', determine the maximum deflection, the maximum bending stress in the beam, and the maximum pressure between the beam and the foundation.

The moment of inertia of the beam about x-axis is

$$I_x = 66.67(10^6)\,\text{mm}^4$$

The elastic coefficient,

$$k = bk_o = 100(0.040) = 4.00\,\text{N/mm}^2$$

and the value of β,

$$\beta = \sqrt[4]{\frac{4}{4(10)10^3(66.67)10^6}} = 0.001107\,\text{mm}^{-1}$$

From the graph of the deflection of the beam as shown in Fig. 8.6b, the maximum deflection occurs at the center of segment L'. Since $a = b = L'/2$,

$$\beta a = \beta b = \beta\frac{L'}{2} = 2.0$$

$$D_{\beta a} = D_{\beta b} = -0.0563$$

$$y_{\max} = \frac{w}{2k}\left[2 - D_{\beta a} - D_{\beta b}\right] = \frac{35}{4}(2 - 2(-0.0563)) = 9.243\,\text{mm}$$

The maximum pressure between the beam and the foundation occurs at the point of the maximum deflection.

$$q_{\max} = k_o y_{\max} = 0.040(9.243) = 0.370\,\text{MPa}$$

They are 4 possible locations at which the maximum bending moment may occur. However, since the beam is symmetry with respect to the center of the segment L', the maximum bending moment may be occurred at the center of the segment L' or where $V_H = 0$.

Since $\beta L' = 0.001107(3.61)10^3 = 4.00 > \pi$, the maximum bending moment does not occur at the center of the segment L'. Hence, L', the maximum bending moment will occur at the location where $V_H = 0$.

$$V_H = \frac{w}{4\beta}\left[C_{\beta a} - C_{\beta b}\right] = 0$$

$$C_{\beta a} = C_{\beta b}$$

$$e^{-\beta a}(\cos\beta a - \sin\beta a) = e^{-\beta b}(\cos\beta b - \sin\beta b)$$

Using the above equation and $\beta b = 4.0 - \beta a$, we can solve the equation and obtain $\beta a = 0.858$ and -0.777. The corresponding $\beta b = 3.142$ and 4.777, respectively. These conditions locate the position of the maximum relative bending moments inside segment L' and outside segment L'.

By comparing the conditions, we can see that the maximum bending moment occurs outside segment L' and

$$M_{\max} = \left| \frac{-w}{4\beta^2}\left[B_{\beta a} - B_{\beta b}\right] \right|$$
$$= \frac{35}{4(0.001107)^2}\left[0.3223 - (-0.0086)\right]$$
$$= 2.363 \text{ kN - m}$$

which is larder than the bending moments occurred inside segment L' by about 3%.

The corresponding bending stress is

$$\sigma_{\max} = \frac{M_{\max} c}{I_x} = 3.544 \text{ MPa}$$

It should be noted that if the maximum bending moment is assumed to occur at $\pi / 4\beta$, $\beta a = \pi / 4$ and $\beta b = 4 - \pi / 4$ (inside segment L'), we obtain the bending moments equal to

$$M_H = \frac{w}{4\beta^2}\left[B_{\beta a} + B_{\beta b}\right]$$
$$= \frac{35}{4(0.001107)^2}\left[0.3224 + (-0.0029)\right]$$
$$= 2.362 \text{ kN - m}$$

8.6 Semi-infinite beam Subjected to Loads at Its End

Consider the semi-infinite beam subjected to a point load P and a positive bending moment M_o at its end as shown in Fig. 8.7. The displacement solution of the beam is in the form of, Eq. 8.7,

$$y = e^{-\beta z}(C_3 \sin\beta z + C_4 \cos\beta z) \qquad z \geq 0$$

In this case the constants of integration C_3 and C_4 can be determined by using the boundary conditions:

$$EI_x \left.\frac{d^2 y}{dz^2}\right|_{z=0} = -M_o$$

$$EI_x \left.\frac{d^3 y}{dz^3}\right|_{z=0} = -V_y = P \qquad 8.13$$

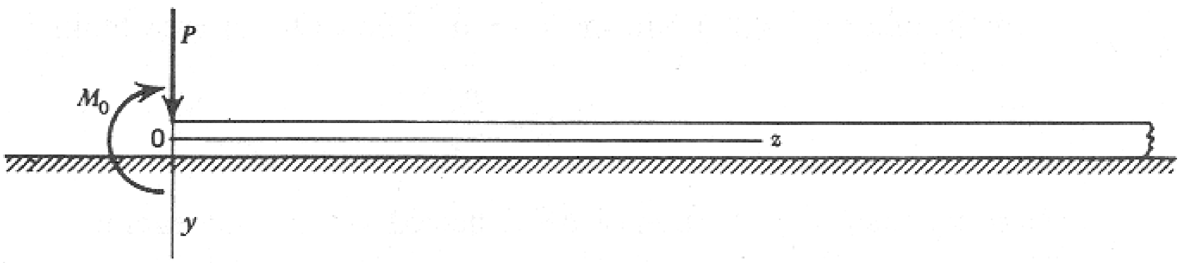


Fig. 8.7

Since $\frac{d^2 y}{dz^2} = -\frac{2\beta^2}{e^{\beta z}}\left[C_3 \cos\beta z - C_4 \sin\beta z\right]$,

$$C_3 = \frac{M_o}{2\beta^2 EI_x} = \frac{2\beta^2 M_o}{k} \qquad 8.14$$

Since $\frac{d^3 y}{dz^3} = \frac{2\beta^3}{e^{\beta z}}\left[C_3 \sin\beta z + C_4 \cos\beta z + C_3 \cos\beta z - C_4 \sin\beta z\right]$,

$$C_3 + C_4 = \frac{P}{2\beta^3 EI_x} = \frac{2\beta P}{k}$$

$$C_4 = \frac{2\beta P}{k} - \frac{2\beta^2 M_o}{k}$$

Thus, the deflection of the beam in this case is

$$y = \frac{2\beta e^{-\beta z}}{k}\left[P\cos\beta z - \beta M_o(\cos\beta z - \sin\beta z)\right] \qquad 8.15$$

Rearranging and simplifying the equation, we have

$$y = \frac{2P\beta}{k} D_{\beta z} - \frac{2\beta^2 M_o}{k} C_{\beta z}$$

The expressions of the slope, shear force, and bending moment can be found by using the differential relation.

$$\theta = -\frac{2P\beta^2}{k} A_{\beta z} + \frac{4\beta^3 M_o}{k} D_{\beta z}$$

$$M_x = -\frac{P}{\beta} B_{\beta z} + M_o A_{\beta z}$$

$$V_y = -PC_{\beta z} - 2M_o \beta B_{\beta z}$$

Example 8-4

A steel I-beam ($E = 200\,\text{GPa}$) has a depth of $102\,\text{mm}$, a width of $68\,\text{mm}$, a moment of inertia of $I_x = 2.53(10^6)\,\text{mm}^4$, and a length of $4\,\text{m}$. It is attached to a rubber foundation for which $k_o = 0.350\,\text{N/mm}^3$. A concentrated load $P = 30.0\,\text{kN}$ is applied at one end of the beam. Determine the maximum deflection, the maximum bending stress in the beam, and their locations.

The spring coefficient,

$$k = 68(0.350) = 23.8\,\text{N/mm}^2$$

and the value of β,

$$\beta = \sqrt[4]{\frac{23.8}{4(200)10^3(2.53)10^6}} = 0.001852\,\text{mm}^{-1}$$

Since

$$L = 4000\,\text{mm} > \frac{3\pi}{2\beta} = 2540\,\text{mm}$$

the beam can be considered as a long beam.

The maximum deflection occurs at the end where load P is applied ($z = 0$), since $D_{\beta z}$ is maximum. We have $\beta z = 0$ and $D_{\beta z} = 1.0$.

$$y_{\max} = \frac{2P\beta}{k} D_{\beta z} = \frac{2(30)10^3(0.001852)}{23.8}(1) = 4.67\,\text{mm}$$

The maximum bending occurs at $z = \pi / 4\beta$, where $B_{\beta z}$ is maximum. This is the same location of the maximum bending stress.

$$M_{\max} = -\frac{P}{\beta} B_{\beta z} = -\frac{30(10^3)}{0.001852}(0.3224) = -5.22\,\text{kN - m}$$

and

$$\sigma_{\max} = 105.3\,\text{MPa}$$

Chapter 9

Flat Plates

9.1 Introduction

Flat plate is a structural element whose middle surface lies in a flat plane and subjected to lateral load q. Floor slabs and pavements are the common examples. The plate can be categorized according to its thickness relative to its other dimensions and according to its lateral deflection compared to its thickness.

1. Relatively thick plates with small deflections
2. Relatively thin plates with small deflections
3. Very thin plate with large deflection
4. Extremely thin plates or membrane

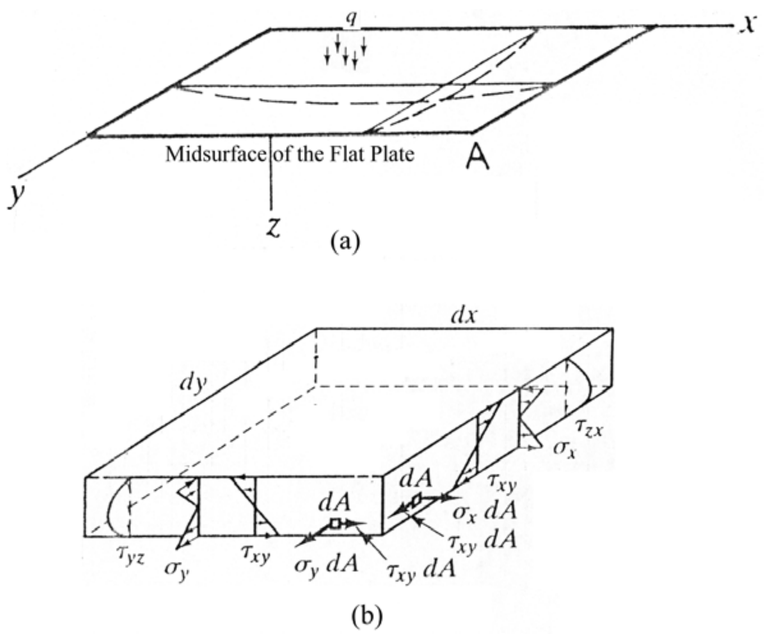


Fig. 9.1

Under the action of the lateral load q, the midsurface of the flat plate is deflected as shown in Fig. 9.1a. Fig. 9.1b shows the state of stresses and their distribution in a small element of the plate. The governing equations of the flat plate can be determined by using the equilibrium equations, the strain-displacement relations, and the stress-strain relations.

9.2 Assumptions and Limitations of Thin Plate with Small Deflection

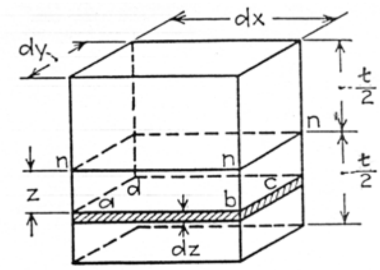
In the classical thin-plate theory or Kirchhoff theory, the following assumptions are applied:

1. The plate is flat and has a constant thickness.
2. The plate has a relatively small thickness compared to the smallest lateral dimensions.

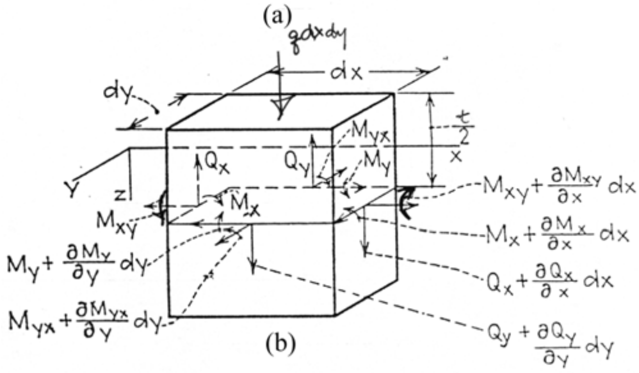
3. The plate is made of linearly elastic, isotropic, homogeneous materials.
4. The plate is subjected to lateral loads applied perpendicular to the midsurface.
5. The midsurface deflection w is small in comparison with its thickness t.
6. The midsurface remains neutral during loading. There is no deformation in the midsurface of the plate. The strain ε_x, ε_y, and γ_{xy} are zero at $z = 0$.
7. A line normal to the midsurface before loading remains normal to the midsurface after loading. The transverse shear strain γ_{yz} and γ_{xz} are zero.
8. The normal stress in the direction transverse to the plate σ_z is negligible in comparison with σ_x and σ_y.

Following the assumptions, the lateral deflection of the plate is a function of only coordinate x and y, and the stresses σ_x, σ_y, and τ_{xy} are linearly distributed as shown in Fig. 9.1b.

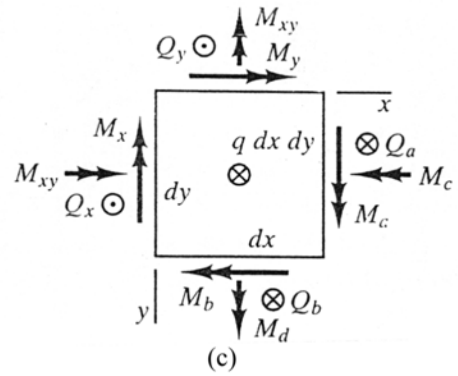
9.3 Force-Stress Relations



(a)



(b)



(c)

Fig. 9.2

Consider a plate differential element as shown in Fig. 9.1 and Fig. 9.2. The increment of the bending moments dM_x and dM_y, the twisting moment dM_{xy}, and the transverse shear forces dQ_x and dQ_y can be found in the form of

$$dM_x = z(\sigma_x dA)$$

$$dM_y = z(\sigma_y dA)$$

$$dM_{xy} = z(\tau_{xy} dA)$$

$$dQ_x = \tau_{xz} dA$$

$$dQ_y = \tau_{yz} dA \qquad 9.1$$

If we define the bending moments M_x and M_y, the twisting moment M_{xy}, and the transverse shear forces Q_x and Q_y ***per unit length***. Then, the differential area dA is $dA = (1)dz$, and the moments and the shear forces can be found by integrating the corresponding moments and shear forces.

$$M_x = \int_{-t/2}^{t/2} \sigma_x z dz$$

$$M_y = \int_{-t/2}^{t/2} \sigma_y z dz$$

$$M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} z dz$$

$$Q_x = \int_{-t/2}^{t/2} \tau_{xz} dz$$

$$Q_y = \int_{-t/2}^{t/2} \tau_{yz} dz \qquad 9.2$$

9.4 Equilibrium Equations

By using the equilibrium equations on the plate differential element as shown in Fig. 9.2b, we have

$+\downarrow \sum F_z = 0;$

$$-Q_x dy - Q_y dx + qdxdy + \left[Q_x + \frac{\partial Q_x}{\partial x} dx\right]dy + \left[Q_y + \frac{\partial Q_y}{\partial y} dy\right]dx = 0$$

$$\left[\frac{\partial Q_x}{\partial x} dx\right]dy + \left[\frac{\partial Q_y}{\partial y} dy\right]dx + qdxdy = 0$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0 \tag{9.3}$$

$\sum M_x = 0;$

$$-M_{xy}dy - M_y dx + \left[M_{xy}+\frac{\partial M_{xy}}{\partial x}dx\right]dy + \left[M_y+\frac{\partial M_y}{\partial y}dy\right]dx - \left[Q_y+\frac{\partial Q_y}{\partial y}dy\right]dxdy$$

$$-\left[Q_x+\frac{\partial Q_x}{\partial x}dx\right]dy\frac{dy}{2}+Q_x dy\frac{dy}{2}-qdxdy\frac{dy}{2}=0$$

Neglecting the higher-order term, we have

$$\left[\frac{\partial M_{xy}}{\partial x}dx\right]dy+\left[\frac{\partial M_y}{\partial y}dy\right]dx - Q_y dxdy = 0$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=0 \tag{9.4}$$

Similarly, the $\sum M_y = 0$ will provide us with the equation

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0 \tag{9.5}$$

Substituting Eq. 9.4 and 9.5 into Eq. 9.3, we have the equation relating the external lateral load q with the internal resultant bending moments in the form of

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+q=0 \tag{9.6}$$

9.5 Kinetics: Strain-Displacement Relations

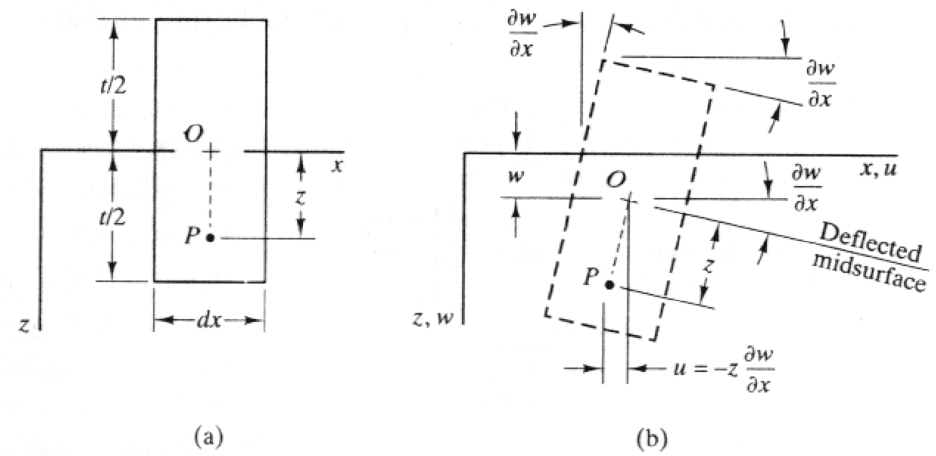


Fig. 9.3

Consider Fig. 9.3 showing the differential slice of the plate viewed parallel to the y axis. The displacement in the x axis of the point P is

$$u=-z\frac{\partial w}{\partial x}$$

If we view the slice of the plate parallel to the x axis, we have the displacement in the y axis of the point P is

$$v = -z\frac{\partial w}{\partial y}$$

Hence, the strains components on the element due to the displacement are

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} \qquad 9.7$$

9.6 Stress-Strain Relations

For linear elastic isotropic homogeneous material,

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)$$

$$\tau_{xy} = G\gamma_{xy} \qquad 9.8$$

where $G = \dfrac{E}{2(1+\nu)}$.

9.7 Stress-Deflection Relations

Substituting Eq. 9.7 into Eq. 9.8, we have

$$\sigma_x = \frac{E}{1-\nu^2}\left[-z\frac{\partial^2 w}{\partial x^2} - \nu z\frac{\partial^2 w}{\partial y^2}\right]$$

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right] \qquad 9.9a$$

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] \qquad 9.9b$$

$$\tau_{xy} = -2Gz\frac{\partial^2 w}{\partial x \partial y} \qquad 9.9c$$

9.8 Governing Differential Equations

Substituting Eq. 9.9 into the internal force-stress relations, Eq. 9.2, we have the expressions for the internal forces and the displacement w.

$$M_x = \int_{-t/2}^{t/2} \sigma_x z dz = \int_{-t/2}^{t/2} -\frac{Ez^2}{1-\nu^2}\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right] dz$$

$$M_x = -\frac{E}{1-\nu^2}\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right]\int_{-t/2}^{t/2} z^2 dz$$

Since $\int_{-t/2}^{t/2} z^2 dz = \frac{t^3}{12}$,

$$M_x = -\frac{Et^3}{12(1-\nu^2)}\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right]$$

Defining $D = \frac{Et^3}{12(1-\nu^2)}$ which is the plate flexural rigidity.

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right] \qquad 9.10a$$

Similarly,

$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right] \qquad 9.10b$$

$$M_{xy} = -2\frac{E}{2(1+\nu)}\frac{\partial^2 w}{\partial x \partial y}\int_{-t/2}^{t/2} z^2 dz = -\frac{Et^3}{12(1+\nu)}\frac{\partial^2 w}{\partial x \partial y}$$

$$M_{xy} = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} \qquad 9.10c$$

Substituting Eq. 9.10 into the equilibrium equation, Eq 9.6, we have the governing equation for the thin flat plate.

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

$$-D\left[\frac{\partial^4 w}{\partial x^4} + \nu \frac{\partial^4 w}{\partial^2 x \partial y^2}\right] - 2(1-\nu)D\frac{\partial^4 w}{\partial x^2 \partial y^2} - D\left[\frac{\partial^4 w}{\partial y^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial^2 y}\right] + q = 0$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D} \qquad 9.11$$

The deflection of the midsurface of the flat plate $w = w(x,y)$ can be determined by integrating this governing equation. Then, the moment expressions are obtained by substituting the deflection expressions into the expressions of the moment, Eq. 9.10.

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right]$$

$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]$$

$$M_{xy} = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y}$$

The transverse shearing force can be obtained from the Eq. 9.4 and 9.5,

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

$$Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}$$

Thus,

$$Q_x = -D\left[\frac{\partial^3 w}{\partial x^3} + \nu\frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial^3 w}{\partial x \partial y^2} - \nu\frac{\partial^3 w}{\partial x \partial y^2}\right] = -D\left[\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right]$$

$$Q_x = -D\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] \qquad 9.12a$$

Similarly,

$$Q_y = -D\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] \qquad 9.12b$$

By assuming that the transverse

9.9 Boundary Conditions

The most frequently encountered boundary conditions for rectangular plates are essentially the same as those for beams. They are either fixed, simply supported, free, or partially fixed as shown in Fig. 9.4.

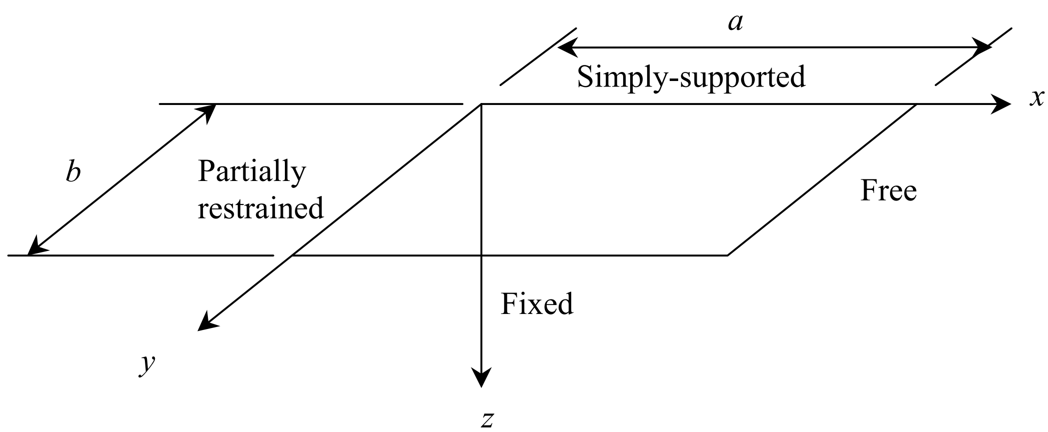


Fig. 9.4

Fixed Edges

For the fixed edge, the deflection and slope are zero. Thus,

$$w\big|_{y=b} = 0$$

$$\left.\frac{\partial w}{\partial y}\right|_{y=b} = 0 \qquad 9.13$$

Simply Supported edge

For simply supported edge, the deflection and moment are zero. Thus,

$$w\big|_{y=0} = 0 \qquad 9.14a$$

$$M_y\big|_{y=0} = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]_{y=0} = 0$$

The expression $\nu\frac{\partial^2 w}{\partial x^2}$ can be rewritten as $\nu\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)$. The term $\frac{\partial w}{\partial x}$ is the rate of change of the slope at the boundary. But, the change in slope along the simply support edge $y = 0$ is always zero. Hence, the quantity $\nu\frac{\partial^2 w}{\partial x^2}$ vanishes and the moment boundary condition is

$$M_y\big|_{y=0} = \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0} = 0 \qquad 9.14b$$

Free Edge

At the free edge, the moment and shear are zero. Thus,

$$M_x\big|_{x=a} = M_{xy}\big|_{x=a} = Q_x\big|_{x=a} = 0$$

$$\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]_{x=a} = 0 \qquad 9.15a$$

The last two boundary conditions can be combined into a single equation. Consider the Fig. 9.5, Kirchhoff has shown that the moment M_{xy} can be though of as a series of couples acting on an infinitesimal section. Hence, at any point along the edge

$$Q' = \left.\frac{\partial M_{xy}}{\partial y}\right|_{x=a}$$

This equivalent shearing force, Q', must be added to the shearing force Q_x acting at the edge. Therefore, the total shearing force is

$$V_x = \left.\left(Q_x + \frac{\partial M_{xy}}{\partial y}\right)\right|_{x=a} = 0 \qquad 9.15b$$

Substituting $Q_x = -D\dfrac{\partial}{\partial x}\left[\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right]$ and $M_{xy} = -(1-\nu)D\dfrac{\partial^2 w}{\partial x \partial y}$ into Eq. 9.15b, we obtain

$$\left.\left(\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right)\right|_{x=a} = 0 \qquad 9.15c$$

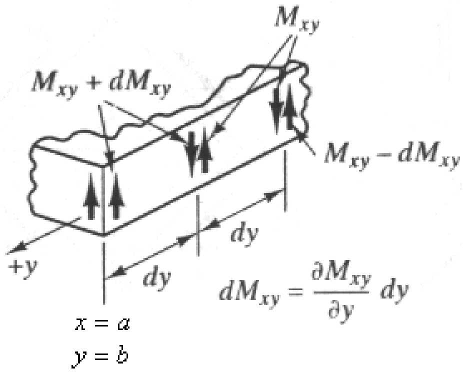


Fig. 9.5

Partially Restrained Edge

A partially restrained edge occurs in when the plate is connected to the beam as shown in Fig. 9.6. In this case, the following boundary conditions must be satisfied.

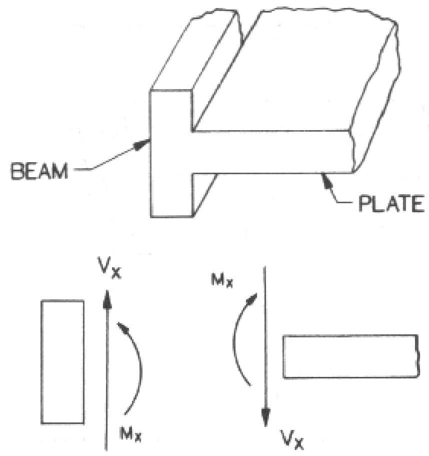


Fig. 9.6

$$V\big|_{plate} = V\big|_{beam}$$

$$-D\left.\left(\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right)\right|_{x=0} = EI\left.\left(\frac{\partial^4 w}{\partial y^4}\right)\right|_{x=0} \qquad 9.16a$$

and

$$M\big|_{plate} = M\big|_{beam}$$

$$-D\left.\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]\right|_{x=0} = GJ\left.\left(\frac{\partial^3 w}{\partial x \partial y^2}\right)\right|_{x=0} \qquad 9.16b$$

Corner Reactions

It was shown in the derivation of the boundary condition of the shearing force at the free edge that the torsion moment M_{xy} as shown in Fig. 9.5 can be resolved into a series of couples. At any corners such as at $x = a$ and $y = b$, the moment M_{xy} results in an upward force R as shown in Fig. 9.7.

$$R = -2M_{xy}\Big|_{\substack{x=a\\y=b}} = 2(1-\nu)D\left(\frac{\partial^2 w}{\partial x \partial y}\right)\Bigg|_{\substack{x=a\\y=b}} \qquad 9.17$$

This equation is usually used to determine the force in corner bolts of rectangular plates.

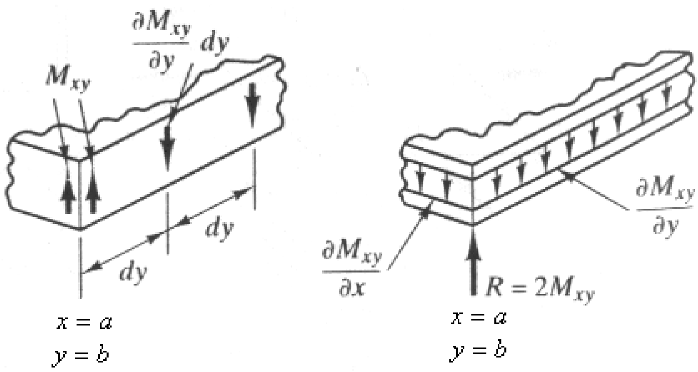


Fig. 9.7

Example 9-1

Determine the moment and reaction for a simply supported rectangular plate of length a in the x direction and width b in the y direction as shown in Fig. Ex 9-1. The plate is subjected to a sinusoidal lateral load, $q = q_o \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$.

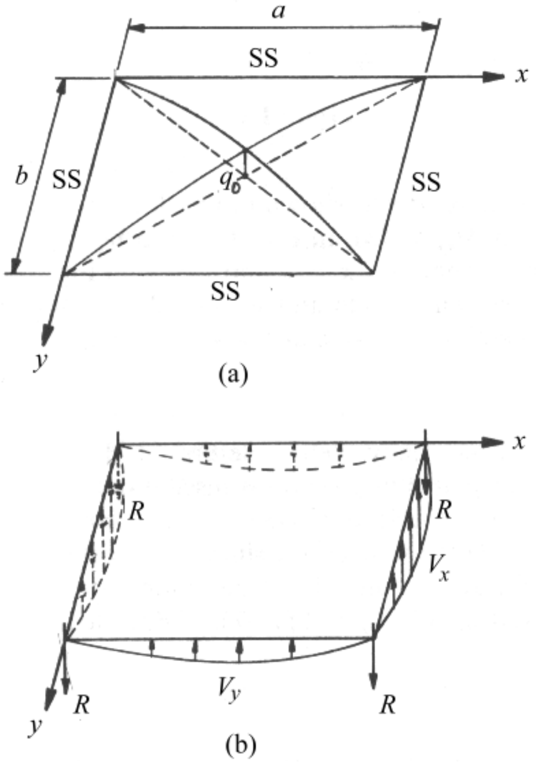


Fig. Ex 9-1

The governing differential equation of the plate, Eq. 9.11, is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q_o}{D}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

For the simply supported plate, the boundary conditions, Eq. 9.14, are

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = 0 \text{ and } x = a$$

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } y = 0 \text{ and } y = b$$

In order to solve the governing differential equation for the deflection, the assumed deflection equation must be in the same form as that of the governing equation and must satisfy the boundary conditions. Thus,

$$w = C\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

Substituting the deflection equation into the governing equation, we obtain

$$C = \frac{q_o}{D\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}$$

Thus, the assumed deflection equation is

$$w = \frac{q_o}{D\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

Substituting this equation into the force-stress relations, Eq. 9.10, we get

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right] = \frac{q_o}{\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}\left(\frac{1}{a^2}+\frac{\nu}{b^2}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right] = \frac{q_o}{\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}\left(\frac{\nu}{a^2}+\frac{1}{b^2}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$M_{xy} = -(1-\nu)D\frac{\partial^2 w}{\partial x\partial y} = -\frac{q_o(1-\nu)}{\pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 ab}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}$$

It should be noted that the maximum M_x and M_y occur at $x = a/2$ and $y = b/2$. Substituting the deflection equation into the transverse shearing forces, Eq. 9.12, we have

$$Q_x = -D\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right] = \frac{q_o}{\pi a\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$Q_y = -D\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right] = \frac{q_o}{\pi b\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}$$

The reaction on edge $x = a$ can be determined by using the Eq. 9.15b,

$$V_x = \left(Q_x + \frac{\partial M_{xy}}{\partial y}\right)\Bigg|_{x=a} = -\frac{q_o}{\pi a\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}\left(\frac{1}{a^2}+\frac{2-\nu}{b^2}\right)\sin\frac{\pi y}{b}$$

The reaction on edge $y = b$ can be determined by using the equation,

$$V_y = \left(Q_y + \frac{\partial M_{xy}}{\partial x}\right)\Bigg|_{y=b} = -\frac{q_o}{\pi b\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}\left(\frac{1}{b^2}+\frac{2-\nu}{a^2}\right)\sin\frac{\pi x}{a}$$

The total reaction around the plate can be determined by integrating the reaction equations from $x=0$ to $x=a$ and from $y=0$ to $y=b$ and then multiplying by 2 due to symmetry.

$$\text{total reaction} = \frac{4q_o ab}{\pi^2} + \frac{8q_o(1-\nu)}{\pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 ab}$$

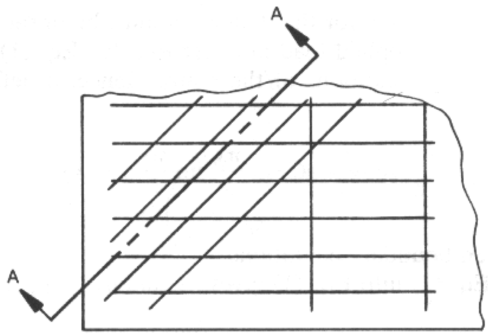
The first part of this equation can also be obtained by integrating the applied load over the total area.

$$\int_0^b\int_0^a q_o \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}dxdy$$

The second part is the summation of the four corner reactions that can be determined by using the Eq. 9.17. Thus, for example,

$$R = -2M_{xy}\Big|_{\substack{x=0\\y=0}} = 2(1-\nu)D\left(\frac{\partial^2 w}{\partial x\partial y}\right)\Bigg|_{\substack{x=0\\y=0}} = \frac{2q_o(1-\nu)}{\pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 ab}$$

The positive value of R at the corner $x=y=0$ means that the reaction force has the downward direction. Thus, it indicates that the corners tend to lift up. This action must be considered when designing the concrete slab. The top corner reinforcements as shown in Fig. Ex 9-1c are needed to resist these forces.



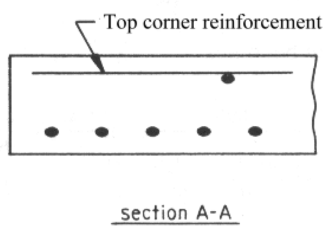


Fig. Ex 9-1c

9.10 Double Series Solution of Simply Supported Plates

Navier has present the solution of a simply supported rectangular plate subjected to uniform load q by representing the load in the double trigonometric series as

$$q(x,y)=\sum_{m=1}^{\alpha}\sum_{n=1}^{\alpha}q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad 9.18$$

To calculate any particular coefficient $q_{m'n'}$ of this series, we multiply both sides of the series by $\sin\frac{n'\pi y}{b}dy$ and integrate from $y=0$ to $y=b$. Then we can see that

$$\int_0^b \sin\frac{n\pi y}{b}\sin\frac{n'\pi y}{b}dy=0 \qquad \text{when } n\neq n'$$

$$\int_0^b \sin\frac{n\pi y}{b}\sin\frac{n'\pi y}{b}dy=\frac{b}{2} \qquad \text{when } n=n'$$

In this way, we find

$$\int_0^b f(x,y)\sin\frac{n'\pi y}{b}dy=\frac{b}{2}\sum_{m=1}^{\alpha}q_{mn'}\sin\frac{m\pi x}{a}$$

Multiplying both sides of this equation by $\sin\frac{m'\pi x}{a}dx$ and integrating from $x=0$ to $x=a$, we obtain

$$\int_0^a\int_0^b f(x,y)\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}dxdy=\frac{ab}{4}q_{m'n'}$$

Thus, the coefficient q_{mn} can be written as

$$q_{mn}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy \qquad 9.19$$

Similarly, the plate deflection w is determined by

$$w(x,y)=\sum_{m=1}^{\alpha}\sum_{n=1}^{\alpha}w_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad 9.20$$

This equation satisfies four boundary conditions of a simply supported plate. The constant w_{mn} can be determined by substituting the plate deflection equation into the governing differential equation of the plate.

Example 9-2

Determine the maximum deflection and bending moments a simply supported plate as shown in Fig. Ex 9-2a due to a uniformly distributed load, q_o.

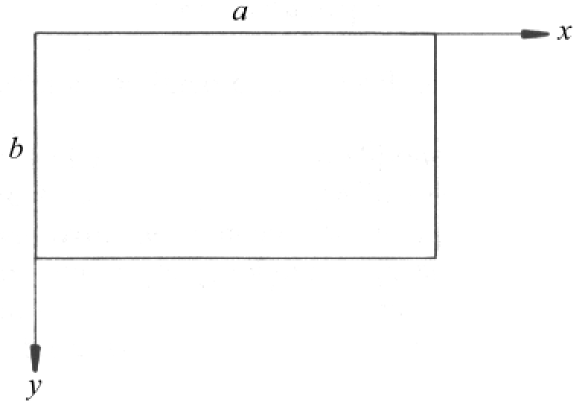


Fig. Ex 9-2a

Since the load is uniformly distributed over the entire plate,

$$f(x,y) = q_o$$

Then, the coefficient q_{mn} is

$$q_{mn} = \frac{4q_o}{ab}\int_0^a\int_0^b \sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}dxdy$$

$$= \frac{4q_o}{\pi^2 mn}(\cos m\pi - 1)(\cos n\pi - 1) = \frac{16q_o}{\pi^2 mn}$$

where $m = 1,3,5,\ldots$ and $n = 1,3,5,\ldots$.

Thus, the uniformly distributed load can be represented by the double trigonometric series as

$$q(x,y) = \frac{16q_o}{\pi^2}\sum_{m=1,3,5,\ldots}^{\alpha}\sum_{n=1,3,5,\ldots}^{\alpha}\frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Substituting this equation and the plate deflection w into the governing differential equation of the plate, $\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$, we obtain the coefficient w_{mn} as

$$w_{mn} = \frac{16q_o}{\pi^6 mnD\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

where $m = 1,3,5,\ldots$ and $n = 1,3,5,\ldots$. Thus, the deflection of the plate is

$$w = \frac{16q_o}{\pi^6 D} \sum_{m=1,3,5,\ldots}^{\alpha} \sum_{n=1,3,5,\ldots}^{\alpha} \frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

In this case, the deflection is symmetric with respect to the axes $x = a/2$ and $y = b/2$. The maximum deflection is occurred at the center of the plate in which $x = a/2$ and $y = b/2$. Then,

$$w_{\max} = \frac{16q_o}{\pi^6 D} \sum_{m=1,3,5,\ldots}^{\alpha} \sum_{n=1,3,5,\ldots}^{\alpha} \frac{(-1)^{\frac{m+n}{2}-1}}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

It should be noted that this series converge rapidly to the exact solution. Substituting the deflection equation into the force-stress relations, Eq. 9.10, we get the bending moments as

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right] = \frac{16q_o}{\pi^4} \sum_{m=1,3,5,\ldots}^{\alpha} \sum_{n=1,3,5,\ldots}^{\alpha} F_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] = \frac{16q_o}{\pi^4} \sum_{m=1,3,5,\ldots}^{\alpha} \sum_{n=1,3,5,\ldots}^{\alpha} G_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$M_{xy} = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} = -\frac{16q_o(1-\nu)}{\pi^4} \sum_{m=1,3,5,\ldots}^{\alpha} \sum_{n=1,3,5,\ldots}^{\alpha} H_{mn} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

where $F_{mn} = \dfrac{\left(\frac{m}{a}\right)^2 + \nu\left(\frac{n}{b}\right)^2}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$

$$G_{mn} = \frac{\nu\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

$$H_{mn} = \frac{1}{ab\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

The maximum bending moments are occurred at the center of the plate in which $x = a/2$ and $y = b/2$. Fig. Ex 9-2b shows a plot of the equations of the bending moments by assuming that $\nu = 0.3$. The figure also shows a plot of the bending moments M_1 and M_2 that are obtained from Mohr's circle along the diagonal of the plate. It should be noted that M_1 becomes negative near the corner of the plate. This is due to the uplift tendency at the corners. This uplift is resisted by the reaction R that causes tension at the top portion of the plate near the corners as mentioned before.

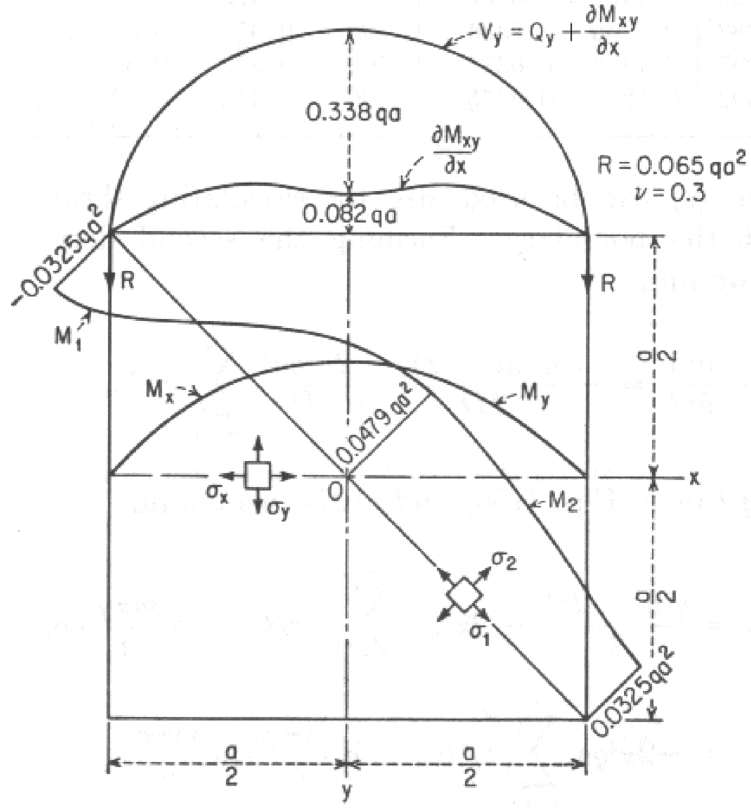


Fig. Ex 9-2b

Finally, substituting $x = a/2$ and $y = b/2$ into the equation of the bending moment, we obtain toe maximum bending moment.

Example 9-3

A simply supported square plate of uniform thickness has the width on each side of $500\,\text{mm}$ as shown in Fig. Ex 9-3. The plate must carry the uniform lateral pressure $q_o = 1.0\,\text{MPa}$ without deflecting more than one-fourth its thickness and without exceeding the allowable normal stress of $250\,\text{MPa}$. Let the material of the plate has $E = 200\,\text{GPa}$ and $\nu = 0.30$ and using the maximum octahedral shearing stress criterion, determine the minimum allowable thickness.

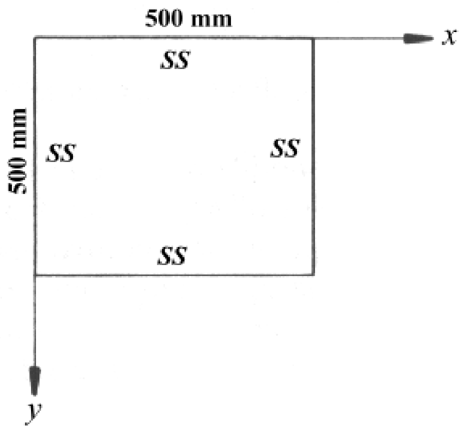


Fig. Ex 9-3

Check the deflection

Due to the symmetry of the plate and loading, the maximum bending moment is occurred at the center of the plate $(x, y) = (0.25, 0.25)\,\text{m}$.

$$w_{\max} = \frac{16q_o}{\pi^6 D} \sum_{m=1,3,5,\ldots}^{\alpha} \sum_{n=1,3,5,\ldots}^{\alpha} \frac{(-1)^{\frac{m+n}{2}-1}}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

Since this series converge rapidly to the exact solution, we will use the first three nonzero series terms for which $m+n$ is smallest which are

$m = 1$ and $n = 1$

$m = 3$ and $n = 1$

$m = 1$ and $n = 3$

$$w_{\max} = \frac{16q_o}{\pi^6 D}\left[\frac{(-1)^{\frac{1+1}{2}-1}}{1(1)\left[\left(\frac{1}{0.5}\right)^2 + \left(\frac{1}{0.5}\right)^2\right]^2} + \frac{(-1)^{\frac{3+1}{2}-1}}{3(1)\left[\left(\frac{3}{0.5}\right)^2 + \left(\frac{1}{0.5}\right)^2\right]^2} + \frac{(-1)^{\frac{1+3}{2}-1}}{1(3)\left[\left(\frac{1}{0.5}\right)^2 + \left(\frac{3}{0.5}\right)^2\right]^2}\right]$$

$$w_{\max} = 256.573(10^{-6})\frac{q_o}{D}$$

The plate stiffness is

$$D=\frac{Et^3}{12(1-\nu^2)}=18.315(10^9)t^3$$

Hence, we have the plate thickness of

$$\frac{t}{4}=256.573(10^{-6})\frac{1(10^6)}{18.315(10^9)t^3}$$

$$t=15.4\text{ mm}$$

Check the allowable normal stress

Due to the symmetry of the plate and loading, the maximum bending moment is occurred at the center of the plate $(x,y)=(0.25,0.25)\text{ m}$ and

$$M_x=M_y=M \text{ and } M_{xy}=0$$

From example 9-2, the maximum bending moment at the center of the plate $(x,y)=(a/2,b/2)$ and the width $a=b$ is

$$M_x=\frac{16q_o}{\pi^4}\sum_{m=1,3,5,\ldots}^{\alpha}\sum_{n=1,3,5,\ldots}^{\alpha}\frac{\left(\frac{m}{a}\right)^2+\nu\left(\frac{n}{b}\right)^2}{mn\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$M_{\max}=\frac{16q_o}{\pi^4}\sum_{m=1,3,5,\ldots}^{\alpha}\sum_{n=1,3,5,\ldots}^{\alpha}\frac{\left(\frac{m}{a}\right)^2+\nu\left(\frac{n}{a}\right)^2}{mn\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{a}\right)^2\right]^2}$$

Since this series converge rapidly to the exact solution, we will use the first three nonzero series terms for which $m+n$ is smallest which are

$$m=1 \text{ and } n=1$$

$$m=3 \text{ and } n=1$$

$$m=1 \text{ and } n=3$$

Thus, we get

$$M_{\max}=\frac{16q_o}{\pi^4}\left[\frac{\left(\frac{1}{0.5}\right)^2+0.3\left(\frac{1}{0.5}\right)^2}{1(1)\left[\left(\frac{1}{0.5}\right)^2+\left(\frac{1}{0.5}\right)^2\right]^2}+\frac{\left(\frac{3}{0.5}\right)^2+0.3\left(\frac{1}{0.5}\right)^2}{3(1)\left[\left(\frac{3}{0.5}\right)^2+\left(\frac{1}{0.5}\right)^2\right]^2}+\frac{\left(\frac{1}{0.5}\right)^2+0.3\left(\frac{3}{0.5}\right)^2}{1(3)\left[\left(\frac{1}{0.5}\right)^2+\left(\frac{3}{0.5}\right)^2\right]^2}\right]$$

$$M_{\max}=11566.3\text{ N-m}$$

The flexural stresses due to the bending moment for unit width of the plate are

$$\sigma_x = \sigma_y = \sigma = \frac{Mc}{I} = \frac{11566.3(t/2)}{1(t^3)/12} = \frac{69398}{t^2}$$

Since $M_{xy} = 0$, τ_{xy} at the center of the plate is zero and the obtained flexural stresses are the principal normal stresses. Using the maximum octahedral shearing stress criterion,

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = 250^2$$

$$2\sigma^2 - \sigma^2 = 250^2$$

$$\frac{69398}{t^2} = 250(10^6)$$

$$t = 16.7 \text{ mm}$$

Since the thickness based on the maximum octahedral shearing stress criterion is larger than the maximum moment condition, determine the allowable thickness is at least

$$t = 16.7 \text{ mm}$$

Example 9-4

Determine the deflection for the simply supported plate subjected to the uniformly distributed load over the area of rectangular as shown in Fig. Ex 9-4.

By virtual of Eq. 9.18 and Eq. 9.19, the coefficient q_{mn} of the uniformly distributed load can be determined from the equation

$$q_{mn} = \frac{4}{abcd} \int_{f-d/2}^{f+d/2} \int_{e-c/2}^{e+c/2} q_o \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dxdy$$

$$q_{mn} = \frac{16 q_o}{mncd\pi^2} \sin\frac{m\pi e}{a} \sin\frac{m\pi c}{2a} \sin\frac{n\pi f}{b} \sin\frac{n\pi d}{2b}$$

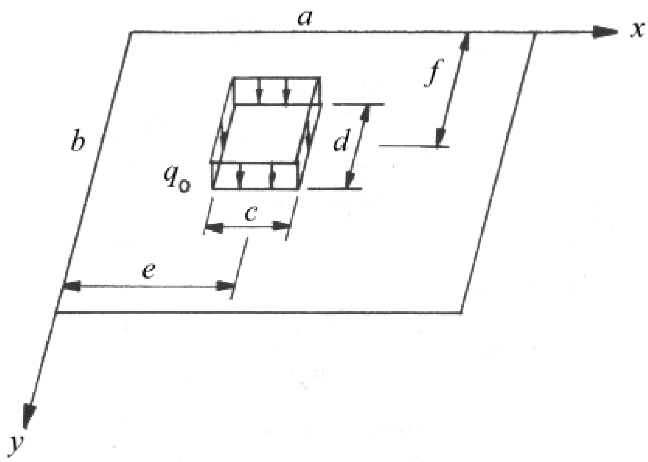


Fig. Ex 9-4

Substituting q_{mn} and the plate deflection w into the governing differential equation of the plate, $\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$, we obtain the coefficient w_{mn} as

$$w_{mn} = \frac{16 q_o}{\pi^6 cdD} \frac{\sin\frac{m\pi e}{a} \sin\frac{m\pi c}{2a} \sin\frac{n\pi f}{b} \sin\frac{n\pi d}{2b}}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

This equation will be reduced to the same equation in the previous example by setting $c = a$, $d = b$, $e = a/2$, and $f = b/2$. Finally, the deflection of the plate can be determined from the equation

$$w = \sum_{m=1,3,\ldots}^{\alpha} \sum_{n=1,3,\ldots}^{\alpha} w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

9.11 Single Series Solution of Simply Supported Plates

Levy in 1900 developed the solution of a simply supported rectangular plate subjected to various loading conditions by using single trigonometric series. He suggested that the solution of the governing equation of the plate can be separated into two parts: homogeneous part and particular part.

$$w = w_h + w_p \tag{9.21}$$

Each of these parts consists of a single trigonometric series where the unknown function is determined from the boundary conditions. The homogeneous part is written as

$$w_h = \sum_{m=1}^{\alpha} f_m(y) \sin \frac{m\pi x}{a} \tag{9.22}$$

where $f_m(y)$ is a function of y only. This equation satisfies the simply supported boundary condition at $x = 0$ and $x = a$.

Substituting w_h into the governing differential equation of the plate, $\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$, we obtain

$$\left[\left(\frac{m\pi}{a}\right)^4 f_m(y) - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 f_m(y)}{dy^2} + \frac{d^4 f_m(y)}{dy^4}\right] \sin \frac{m\pi x}{a} = 0$$

This equation only satisfies when the bracketed term is equal to zero. Thus,

$$\frac{d^4 f_m(y)}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 f_m(y)}{dy^2} + \left(\frac{m\pi}{a}\right)^4 f_m(y) = 0 \tag{9.23}$$

The solution of this differential equation can be expressed as

$$f_m(y) = F_m e^{R_m y} \tag{9.24}$$

Substituting Eq. 9.24 into Eq. 9.23, we have

$$R_m^4 - 2\left(\frac{m\pi}{a}\right)^2 R_m^2 + \left(\frac{m\pi}{a}\right)^4 = 0$$

The roots of this equation are

$$R_m = \pm\frac{m\pi}{a}, \pm\frac{m\pi}{a}$$

Hence, the general solution of the differential equation Eq. 9.23 is

$$f_m(y) = C_{1m} e^{\frac{m\pi y}{a}} + C_{2m} e^{-\frac{m\pi y}{a}} + C_{3m} y e^{\frac{m\pi y}{a}} + C_{4m} y e^{-\frac{m\pi y}{a}}$$

where C_{1m}, C_{2m}, C_{3m}, and C_{4m} are constants. This equation can also be written as

$$f_m(y) = A_m \sinh \frac{m\pi y}{a} + B_m \cosh \frac{m\pi y}{a} + C_m y \sinh \frac{m\pi y}{a} + D_m y \cosh \frac{m\pi y}{a}$$

Therefore, the homogeneous solution, Eq. 9.22, is

$$w_h = \sum_{m=1}^{\alpha}\left[A_m \sinh\frac{m\pi y}{a} + B_m \cosh\frac{m\pi y}{a} + C_m y \sinh\frac{m\pi y}{a} + D_m y \cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a} \qquad 9.25$$

where the constants A_m, B_m, C_m, and D_m are determined by using the boundary conditions of the plate.

The particular solution, w_p, can be expressed in a single trigonometric series as

$$w_p = \sum_{m=1}^{\alpha} k_m(y)\sin\frac{m\pi x}{a} \qquad 9.26$$

This equation also satisfies the simply supported boundary condition at $x = 0$ and $x = a$.

The distributed load q can be expressed as

$$q(x,y) = \sum_{m=1}^{\alpha} q_m(y)\sin\frac{m\pi x}{a} \qquad 9.27$$

where the coefficient $q_m(y)$ is

$$q_m(y) = \frac{2}{a}\int_0^a q(x,y)\sin\frac{m\pi x}{a}\,dx$$

Substituting the equations of the particular solution w_p, Eq. 9.26, and the distributed load q, Eq. 9.27, into the governing differential equation, we obtain

$$\frac{d^4 k_m(y)}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 k_m(y)}{dy^2} + \left(\frac{m\pi}{a}\right)^4 k_m(y) = \frac{q_m(y)}{D} \qquad 9.28$$

Finally, the solution of the governing differential equation is determined by using Eq. 9.25 and the solution of Eq. 9.28.

Example 9-5

The rectangular plate as shown in Fig. Ex 9-5 is subjected to uniformly distributed load q_o. Determine the deflection of the plate.

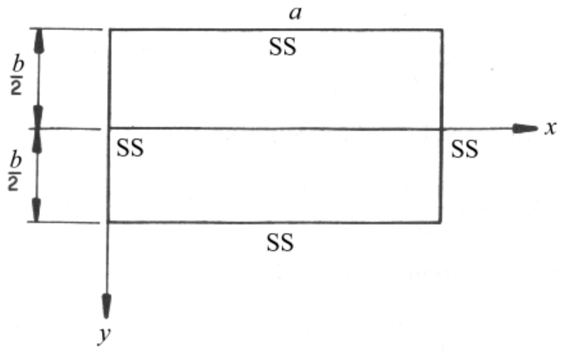


Fig. Ex 9-5

Since $q(x,y)=q_o$ which is a constant, the coefficient $q_m(y)$ of the uniformly distributed load can be expressed as

$$q_m(y)=\frac{2q_o}{a}\int_0^a \sin\frac{m\pi x}{a}dx$$

$$=\frac{2q_o}{m\pi}(-\cos m\pi+1) \quad \text{where } m=1,3,5,\ldots$$

$$=\frac{4q_o}{m\pi}$$

Substituting the distributed load q into Eq. 9.28, we have

$$\frac{d^4k_m(y)}{dy^4}-2\left(\frac{m\pi}{a}\right)^2\frac{d^2k_m(y)}{dy^2}+\left(\frac{m\pi}{a}\right)^4 k_m(y)=\frac{4q_o}{m\pi D}$$

The solution of this differential equation can be taken as $k_m(y)=k_m=$ a constant, and the particular solution $w_p=\sum_{m=1}^{\alpha}k_m\sin\frac{m\pi x}{a}$ satisfies the boundary conditions. Then,

$$k_m=\frac{4a^4q_o}{m^5\pi^5D} \quad \text{where } m=1,3,5,\ldots$$

The particular solution for the deflection of the plate is

$$w_p=\frac{4a^4q_o}{\pi^5D}\sum_{m=1,3,\ldots}^{\alpha}\frac{1}{m^5}\sin\frac{m\pi x}{a}$$

The homogeneous solution for the deflection of the plate is obtained from Eq. 9.25.

$$w_h=\sum_{m=1}^{\alpha}\left[A_m\sinh\frac{m\pi y}{a}+B_m\cosh\frac{m\pi y}{a}+C_m y\sinh\frac{m\pi y}{a}+D_m y\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}$$

By observing the deflection of the plate due to the uniform load, we can see that the deflection in the y direction is symmetric about the x axis. Thus, the constants A_m and D_m

must be set to zero since the quantities $\sinh\dfrac{m\pi y}{a}$ and $y\cosh\dfrac{m\pi y}{a}$ are odd functions as y varies from positive to negative.

In addition, m must be set to $1,3,5,\ldots$ in order for the term $\sin\dfrac{m\pi x}{a}$ to be symmetric about $x = a/2$. Hence,

$$w_h = \sum_{m=1,3,\ldots}^{\alpha}\left[B_m \cosh\frac{m\pi y}{a} + C_m y\sinh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}$$

The total deflection of the plate is

$$w = \sum_{m=1,3,\ldots}^{\alpha}\left[B_m \cosh\frac{m\pi y}{a} + C_m y\sinh\frac{m\pi y}{a} + \frac{4a^4 q_o}{m^5\pi^5 D}\right]\sin\frac{m\pi x}{a}$$

The boundary conditions along the y axis are

$$w = 0 \text{ at } y = \pm b/2$$

$$M_y = 0 \text{ or } \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } y = \pm b/2$$

Then, we have

$$B_m \cosh\frac{m\pi b}{2a} + C_m \frac{b}{2}\sinh\frac{m\pi b}{2a} + \frac{4a^4 q_o}{m^5\pi^5 D} = 0$$

$$\left[B_m\left(\frac{m\pi}{a}\right) + bC_m\right]\cosh\frac{m\pi b}{2a} + C_m\left(\frac{m\pi b}{2a}\right)\sinh\frac{m\pi b}{2a} = 0$$

Solving these two simultaneous equations, we obtain

$$C_m = \frac{2a^3 q_o}{m^4\pi^4 D\cosh\dfrac{m\pi b}{2a}}$$

$$B_m = \frac{4a^4 q_o + m\pi q_o a^3 b\tanh\dfrac{m\pi b}{2a}}{m^5\pi^5 D\cosh\dfrac{m\pi b}{2a}}$$

$$w = \sum_{m=1,3,\ldots}^{\alpha}\left[\begin{array}{l}\dfrac{4a^4 q_o + m\pi q_o a^3 b\tanh\dfrac{m\pi b}{2a}}{m^5\pi^5 D\cosh\dfrac{m\pi b}{2a}}\cosh\dfrac{m\pi y}{a} \\ + \dfrac{2a^3 q_o}{m^4\pi^4 D\cosh\dfrac{m\pi b}{2a}}y\sinh\dfrac{m\pi y}{a} + \dfrac{4a^4 q_o}{m^5\pi^5 D}\end{array}\right]\sin\frac{m\pi x}{a}$$

The maximum deflection is obtained at the center of the plate $x = a/2$ and $y = 0$.

Chapter 10

Buckling and Instability

10.1 Introduction

The selection of structural members is based on three characteristics:

1. strength
2. stiffness
3. stability

Structural instability can occur in numerous situations where the compressive stresses are present. For example,

Long and slender columns subjected to axial compression can buckle long before the material reach the ultimate compressive strength.

Thin-walled tubes can wrinkle when subjected to axial compression.

Narrow beams, unbraced laterally, can turn sidewise and collapse under transverse loads.

Vacuum tank can severely distort under external pressure.

The structural instability and buckling failures are occurred suddenly and dangerous. For the structural members as shown in Fig. 10.1, we can classify the buckling modes as followings:

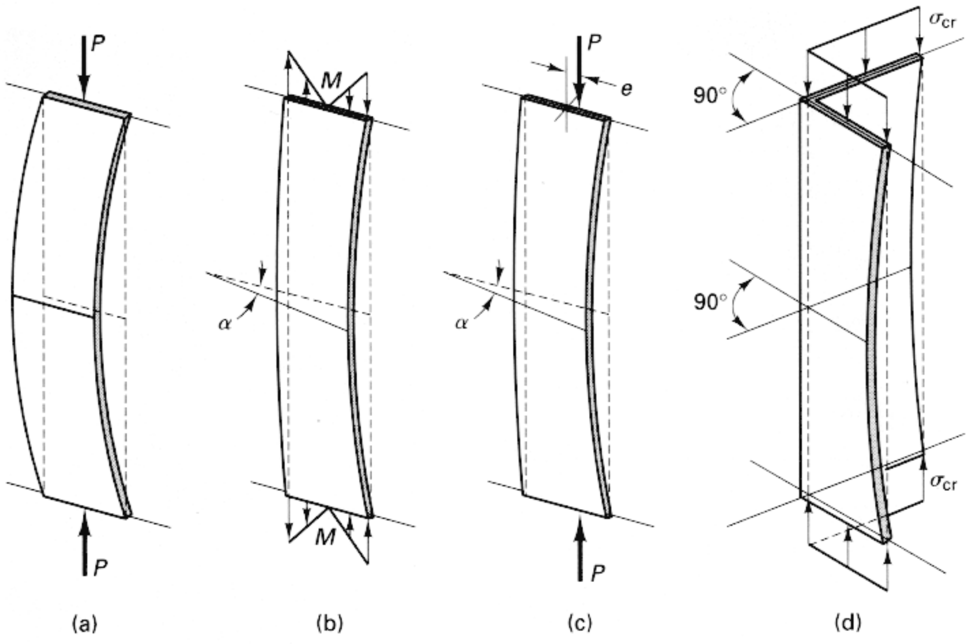


Fig. 10.1

a.) For the column that has limited flexural stiffness but adequate torsional stiffness subjected to compressive force, the dominant buckling mode is the flexural buckling.

b.) For the same column in a.) subjected to bending moment, the dominant buckling mode is the flexural-torsional buckling.

c.) For the same column in a.) subjected to eccentric axial force, the dominant buckling mode is the flexural-torsional buckling.

d.) For the column that has limited torsional stiffness but adequate flexural stiffness subjected to compressive force, the dominant buckling mode is the torsional buckling.

10.2 Column Buckling

Consider an ideal perfectly straight column with pinned supports at both ends as shown in Fig. 10.2. The column is subjected to axially concentric compressive force P and deformed as shown. The bending moment due to the axial force P is $M = -Pv$.

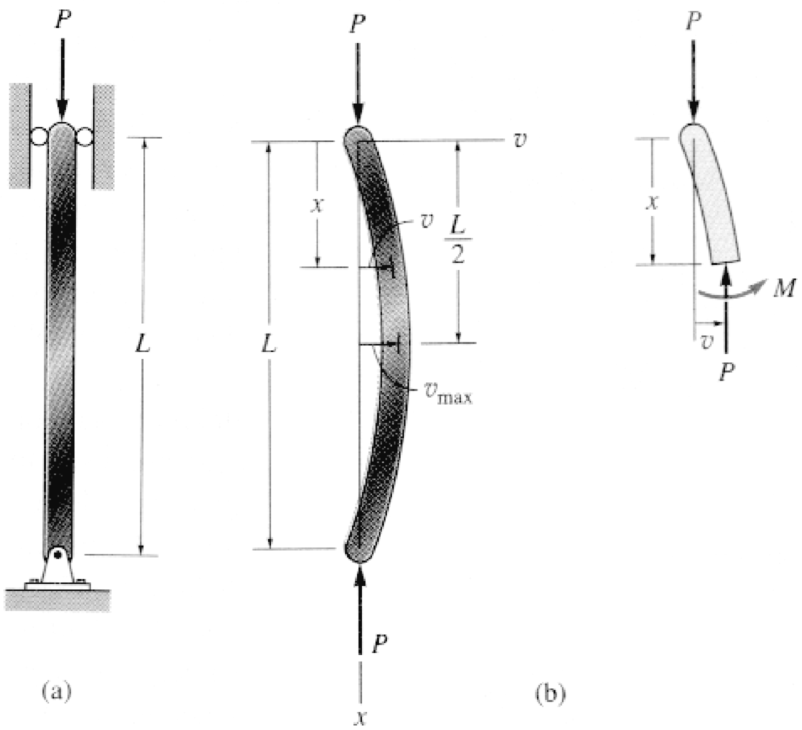


Fig. 10.2

The differential equation for the elastic curve of the column is

$$EI\frac{d^2v}{dx^2} = M$$

$$M = -Pv$$

$$EI\frac{d^2v}{dx^2} = -Pv$$

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v = 0$$

This is the homogenous, linear, differential equation of second order with constant coefficients. It can be solved by assuming that $k^2 = \frac{P}{EI}$. Then,

$$v'' + k^2 v = 0$$

The solution of this equation is in the form of

$$v = C_1 \sin kx + C_2 \cos kx$$

The constants of integration C_1 and C_2 can be determined by using the boundary conditions.

At $x = 0$, $v = 0$; $C_2 = 0$

$$v = C_1 \sin kx$$

At $x = L$, $v = 0$; $C_1 \sin kL = 0$

This condition is satisfied either $C_1 = 0$ or $\sin kL = 0$.

If the constant $C_1 = 0$, the term $\sin kL \neq 0$. Then, the term kL can have any values and the load P can also be any values since $P = k^2(EI)$. Thus, $C_1 = 0$ is the trivial solution.

If the term $\sin kL = 0$ and $C_1 \neq 0$, then, $kL = 0, \pi, 2\pi, 3\pi, \ldots\ldots$ When the term $kL = 0$ (or $k = 0$), the critical load $P = k^2(EI) = 0$ Therefore,

$$kL = n\pi \qquad n = 1,\ 2,\ 3, \ldots$$

$$P = \frac{n^2 \pi^2 EI}{L^2} \qquad n = 1,\ 2,\ 3, \ldots$$

The least force at which a buckled mode is possible is occurred when $n = 1$ and called the critical or Euler buckling load.

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

where $P_{cr} =$ critical bucking load. $P_{cr} < P_y$

$E =$ modulus of elasticity

$I =$ moment of inertia

$L =$ length of the column

Consider a column subjected to axial compressive load. If the column is so slender that its material is always linear elastic until the critical load is reached at point B as shown in Fig. 10.3a, the column can behave into two possible ways when subjected to an increasing axial compressive load. If the column is an ideal column, the column may remain straight (path BC). If the column has a slight imperfection, the column may bend (path BD or path BF) depending on the analytical approaches. If the column has a larger imperfection, the response of the column will follow the path OE.

The somewhat slender column behaves similar to the very slender column as shown in Fig. 10.3b. However, the material of the column will reach the yielding strength, as the deflection grows larger and larger as shown by the downward curve OBD. If the column has a larger imperfection, the response of the column will follow the path OE.

For a lesser slender column, the capacity of the column will be influenced by the yielding strength of the material.

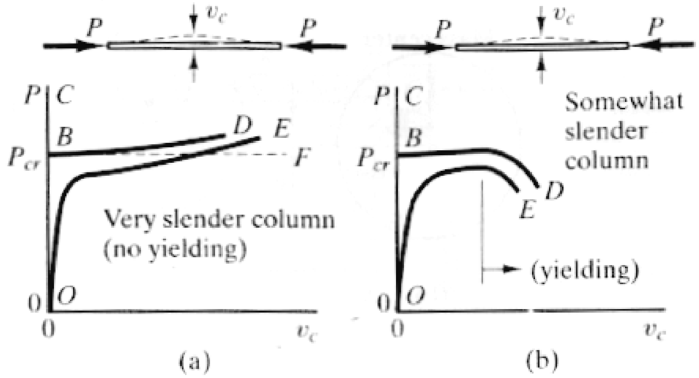


Fig. 10.3

10.3 Plate Buckling

The differential equation of a rectangular plate subjected to lateral load q is obtained in previous chapter as

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

Plates Subjected to Combined Bending and In-plane Loads

If the plate is additionally loaded in its plane by the compression as shown in the Fig. 10.4, summation of forces in the x direction gives

$$\left[N_x + \frac{\partial N_x}{\partial x}dx\right]dy + \left[N_{yx} + \frac{\partial N_{yx}}{\partial y}dy\right]dx - N_x dy - N_y dx = 0$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0$$

Similarly, summation of forces in the y direction provides

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0$$

In considering the forces in the in the z direction, we must take into account the deflection of the plate. Due to the curvature of the plate in the xz plane, the projection of the normal forces N_x on the z axis is

$$-(N_x dy)\frac{\partial w}{\partial x} + \left[N_x + \frac{\partial N_x}{\partial x}dx\right]\left[\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}dx\right]dy$$

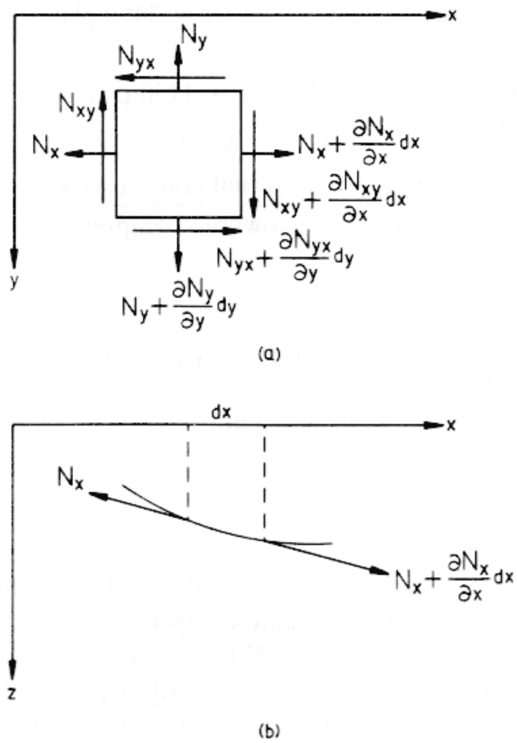


Fig. 10.4

By neglecting the higher order terms, we have

$$N_x \frac{\partial^2 w}{\partial x^2} dxdy + \frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} dxdy$$

Similarly, the projection of the normal forces N_y on the z axis is

$$N_y \frac{\partial^2 w}{\partial y^2} dxdy + \frac{\partial N_y}{\partial y} \frac{\partial w}{\partial y} dxdy$$

Due to the shearing forces N_{xy} and $N_{xy} + \frac{\partial N_{xy}}{\partial x} dx$, the midsurface of the plate is deformed as shown in Fig. 10.5. Owing to the angle $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial x \partial y} dx$, the shearing forces N_{xy} have a projection on the z axis as

$$-\left(N_{xy} dy \frac{\partial w}{\partial y} \right) + \left[N_{xy} + \frac{\partial N_{xy}}{\partial x} dx \right] \left[\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial x \partial y} dx \right] dy$$

$$N_{xy} \frac{\partial^2 w}{\partial x \partial y} dxdy + \frac{\partial N_{xy}}{\partial x} \frac{\partial w}{\partial y} dxdy$$

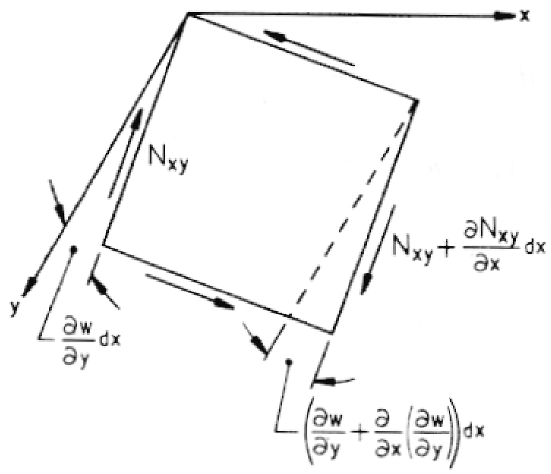


Fig. 10.5

Similarly for N_{yx}

$$N_{yx} \frac{\partial^2 w}{\partial x \partial y} dxdy + \frac{\partial N_{yx}}{\partial y} \frac{\partial w}{\partial x} dxdy$$

Since $N_{yx} = N_{xy}$, the final expression for the projection of the shearing forces on the z axis is

$$2N_{xy} \frac{\partial^2 w}{\partial x \partial y} dxdy + \frac{\partial N_{xy}}{\partial x} \frac{\partial w}{\partial y} dxdy + \frac{\partial N_{yx}}{\partial y} \frac{\partial w}{\partial x} dxdy$$

The total summation of the forces in the z axis is

$$N_x \frac{\partial^2 w}{\partial x^2} dxdy + \frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} dxdy + N_y \frac{\partial^2 w}{\partial y^2} dxdy + \frac{\partial N_y}{\partial y} \frac{\partial w}{\partial y} dxdy$$

$$+ 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} dxdy + \frac{\partial N_{xy}}{\partial x} \frac{\partial w}{\partial y} dxdy + \frac{\partial N_{yx}}{\partial y} \frac{\partial w}{\partial x} dxdy + qdxdy = 0$$

By using the equations $\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0$ and $\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0$, we have

$$q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}$$

The differential equation of a rectangular plate subjected to the combined bending and in-plane loads is obtained by substituting the total summation of the forces in the z axis into the differential equation of a rectangular plate subjected to bending. Then,

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left(q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}\right)$$

Strain Energy in Bending of Plates

The strain energy density of isotropic material is

$$U_o = \frac{1}{2}\left[\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}\right]$$

Thus, the strain energy of a small plate element is

$$U = \frac{1}{2}\int_{Vol}\left[\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy}\right] dxdydz$$

Substituting the stress-deflection relations,

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]$$

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]$$

$$\tau_{xy} = -2Gz\frac{\partial^2 w}{\partial x \partial y}$$

and the strain-deflection relations,

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = -z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

into the equation of the strain energy of a small plate element, we have

$$U = \frac{D}{2}\int\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}dxdy$$

$$U = \frac{D}{2}\int_{Area}\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}dxdy$$

Strain Energy Due to In-plane Loads of Plates

The strain energy due to in-plane loads of plates is derived from Fig. 10.6, which shows the deflection of a unit segment dx. Hence,

$$dx' = \sqrt{dx^2 - \left(\frac{\partial w}{\partial x}dx\right)^2}$$

$$\delta_x = dx - dx' = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx$$

or for a unit length

$$\varepsilon_x = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

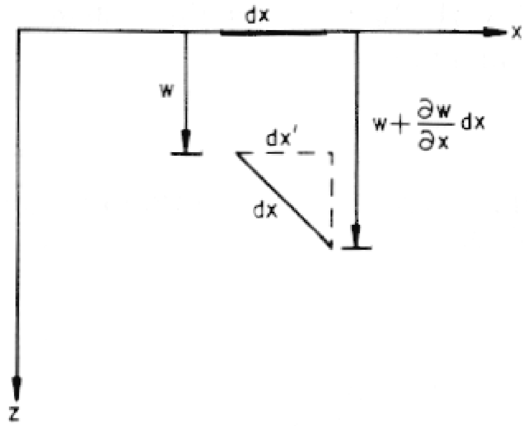


Fig. 10.6

Similarly,

$$\varepsilon_y = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

It can also shown that

$$\gamma_{xy} = \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

Thus, the strain energy due to the in-plane forces is

$$U = \int_{Area} \left[N_x \varepsilon_x + N_y \varepsilon_y + N_{xy}\gamma_{xy} \right] dxdy$$

$$U = \frac{1}{2}\int_{Area} \left[N_x \left(\frac{\partial w}{\partial x}\right)^2 + N_y \left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \right] dxdy$$

Strain Energy Due to Bending and In-plane Loads of Plates

The total strain energy due to bending and in-plane loads is

$$U = \frac{D}{2}\int_{Area} \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dxdy$$

$$+ \frac{1}{2}\int_{Area} \left[N_x \left(\frac{\partial w}{\partial x}\right)^2 + N_y \left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \right] dxdy$$

The total potential energy of the plate is

$$V = \Omega + U$$

In order for the plate to be in equilibrium, the total potential energy of the plate must be minimum.

Example 10-1

Find the buckling stress of a simply supported rectangular plate subjected to the force N_x as shown in Fig. Ex 10-1a.

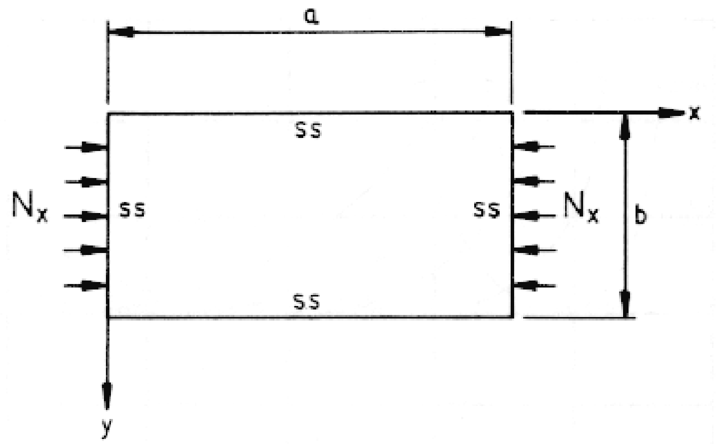


Fig. Ex 10-1a

Let the deflection of the plate be expressed as

$$w(x,y)=\sum_{m=1}^{\alpha}\sum_{n=1}^{\alpha}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

which satisfies the boundary condition of the plate.

Substituting the deflection equation into the total strain energy and noting that the term $2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]=0$, we have

$$U=\frac{D}{2}\int_0^b\int_0^a\sum_{m=1}^{\alpha}\sum_{n=1}^{\alpha}\left[A_{mn}^2\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}\right]dxdy$$

$$+\frac{1}{2}\int_0^b\int_0^a(-N_x)\sum_{m=1}^{\alpha}\sum_{n=1}^{\alpha}\left[A_{mn}^2\left(\frac{m^2\pi^2}{a^2}\right)\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}\right]dxdy$$

or

$$U=\frac{\pi^4 ab}{8}D\sum_{m=1}^{\alpha}\sum_{n=1}^{\alpha}\left[A_{mn}^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2\right]-\frac{\pi^2 b}{8a}N_x\sum_{m=1}^{\alpha}\sum_{n=1}^{\alpha}\left[m^2A_{mn}^2\right]$$

Since there are no lateral loads,

$$V=U$$

For the plate to be in equilibrium, $\frac{\partial V}{\partial A_{mn}}=0$. Then, we have

$$N_{x,cr}=\frac{\pi^2a^2D}{m^2}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2$$

The smallest value of the compressive forces $N_{x,cr}$ is occurred when $n=1$. The physical meaning of this is that a plate buckles in such a way that there can be several half-waves in the direction of compression but only one half-wave in the perpendicular direction. Thus,

$$N_{x,cr} = \frac{\pi^2 D}{a^2}\left(m + \frac{1}{m}\frac{a^2}{b^2}\right)^2$$

If we substituting $\sigma_{cr} = \frac{N_{cr}}{t}$ and $D = \frac{Et^3}{12(1-\nu^2)}$, we get the critical stress in the form of

$$\sigma_{cr} = \frac{\pi^2 E}{12(1-\nu^2)\left(\frac{b}{t}\right)^2} K$$

where $K = \left(\frac{m}{a/b} + \frac{a/b}{m}\right)^2$. The plot of the critical stress is shown in Fig. Ex 10-1b and shows that the minimum value of K is 4.0.

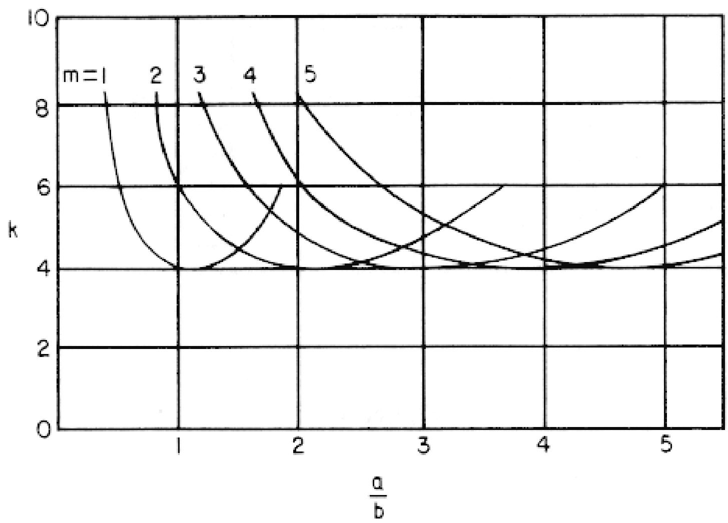


Fig. Ex 10-1b

10.4 Plates with Various Boundary Conditions

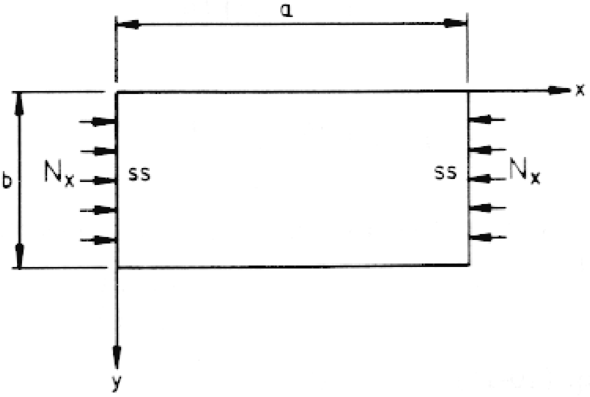


Fig. 10.7

Fig. 10.7 shows a rectangular plate simply supported on sides $x = 0$ and $x = a$ and subjected to the axial compression N_x. The governing differential equation of this plate is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{N_x}{D}\frac{\partial^2 w}{\partial x^2}$$

Let the deflection solution of this plate is in the form of

$$w(x, y) = \sum_{m=1}^{\alpha} f(y)\sin\frac{m\pi x}{a}$$

This solution satisfies the two boundary conditions $w = \frac{\partial^2 w}{\partial x^2} = 0$ at $x = 0$ and $x = a$.

Substituting the deflection solution into the governing differential equation, we obtain

$$\frac{d^4 f}{dy^4} - A\frac{d2f}{dy^2} + Bf = 0$$

where $A = \frac{2m^2\pi^2}{a^2}$

$$B = \frac{m^4\pi^4}{a^4} - \frac{N_x}{D}\frac{m^2\pi^2}{a^2}$$

The general solution of this fourth order differential equation is

$$f(y) = C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3 \cos\beta y + C_4 \sin\beta y$$

where $\alpha = \sqrt{\frac{m^2\pi^2}{a^2} + \sqrt{\frac{N_x}{D}\frac{m^2\pi^2}{a^2}}}$

$$\beta = \sqrt{-\frac{m^2\pi^2}{a^2} + \sqrt{\frac{N_x}{D}\frac{m^2\pi^2}{a^2}}}$$

The values of the constants C_1 through C_4 are obtained from the boundary conditions $y = 0$ and $y = b$.

Case 1

Side $y=0$ is fixed and side $y=b$ is free.

At $y=0$, the deflection $w=0$ and the rotaion $\frac{\partial w}{\partial y}=0$.

At $y=b$, the moment $M_y=\left[\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right]=0$ and the shear

$$Q=\left(\frac{\partial^3 w}{\partial y^3}+(2-\nu)\frac{\partial^3 w}{\partial x\partial y^2}\right)=0$$

From the first boundary condition, we obtain

$$C_1+C_2+C_3=0$$

From the second boundary condition, we obtain

$$-\alpha C_1+\alpha C_2+\beta C_3=0$$

or

$$C_1=-\frac{C_3}{2}+\frac{\beta C_4}{2\alpha}$$

and

$$C_2=-\frac{C_3}{2}-\frac{\beta C_4}{2\alpha}$$

Substituting C_1 and C_2 into $f(y)=C_1e^{-\alpha y}+C_2e^{\alpha y}+C_3\cos\beta y+C_4\sin\beta y$, we get

$$f(y)=C_3(\cos\beta y-\cosh\alpha y)+C_4(\sin\beta y-\frac{\beta}{\alpha}\sinh\alpha y)$$

Substituting $f(y)$ into the deflection equation, we have

$$w(x,y)=\sum_{m=1}^{\alpha}\left[C_3(\cos\beta y-\cosh\alpha y)+C_4(\sin\beta y-\frac{\beta}{\alpha}\sinh\alpha y)\right]\sin\frac{m\pi x}{a}$$

Using the last two boundary conditions, we obtain two simultaneous equations. The critical value of the compressive force, N_x, is determined by equating the determinant of these equations to zero.

$$2gh(g^2+h^2)\cos\beta b\cosh\beta b=\frac{1}{\alpha\beta}(\alpha^2h^2-\beta^2g^2)\sin\beta b\sinh\beta b$$

where $g=\alpha^2-\nu\frac{m^2\pi^2}{a^2}$ and $h=\beta^2+\nu\frac{m^2\pi^2}{a^2}$. For $m=1$, the minimum value of the critical compressive stress is

$$\sigma_{cr} = \frac{\pi^2 E}{12(1-\nu^2)\left(\frac{b}{t}\right)^2} K$$

For $\nu = 0.25$,

$$K_{\min} = 1.328$$

Case 2

Side $y = 0$ is simply supported and side $y = b$ is free.

Similarly, in this case, the maximum value of K is

$$K = 0.456 + \frac{b^2}{a^2} \qquad \text{for} \qquad \nu = 0.25$$

Case 3

Side $y = 0$ and side $y = b$ are fixed. In this case, the maximum value of K is

$$K = 7.0 \qquad \text{for} \qquad \nu = 0.25$$

Example 10-2

Let the plate in Fig. Ex 10-1a be simply supported at $x = 0$ and $x = a$, simply supported at $y = 0$, and free at $y = b$. Determine the required thickness if $a = 0.560\,\text{m}$, $b = 0.430\,\text{m}$, $N_x = 52500\,\text{N/m}$, $\nu = 0.25$, $\sigma_y = 250\,\text{MPa}$, $E = 200\,\text{GPa}$, and factor of safety = 2.0.

Assuming that $t = 6.5\,\text{mm}$. From case 2, the value of $K = 1.046$. Then, the critical compressive stress is

$$\sigma_{cr} = \frac{\pi^2(200000)}{12(1-0.25^2)\left(\dfrac{0.430}{0.0065}\right)^2}(1.046) = 41.93\,\text{MPa}$$

which is significantly less than the yielding stress $\sigma_y = 250\,\text{MPa}$.

Thus, the allowable stress is $41.9/2 = 20.95\,\text{MPa}$.

The actual applied stress is $52500/0.0065 = 8.07\,\text{MPa}$. $\therefore$ O.K.

10.5 Application of Buckling to Design Problems

The AISC assumes the buckling stress of unsupported members in compression not to exceed the yielding strength of the material. Thus,

$$\sigma_y = \frac{\pi^2 E}{12(1-\nu^2)\left(\frac{b}{t}\right)^2} K$$

This equation is based on the assumption that the interaction between the buckling stress and the yielding stress is designated by the curve ABC as shown in Fig. 10.8. For steel members with $\nu = 0.3$ and $E = 29000\,\text{ksi}$, we have

$$\frac{b}{t} = 162\sqrt{\frac{K}{\sigma_y}}$$

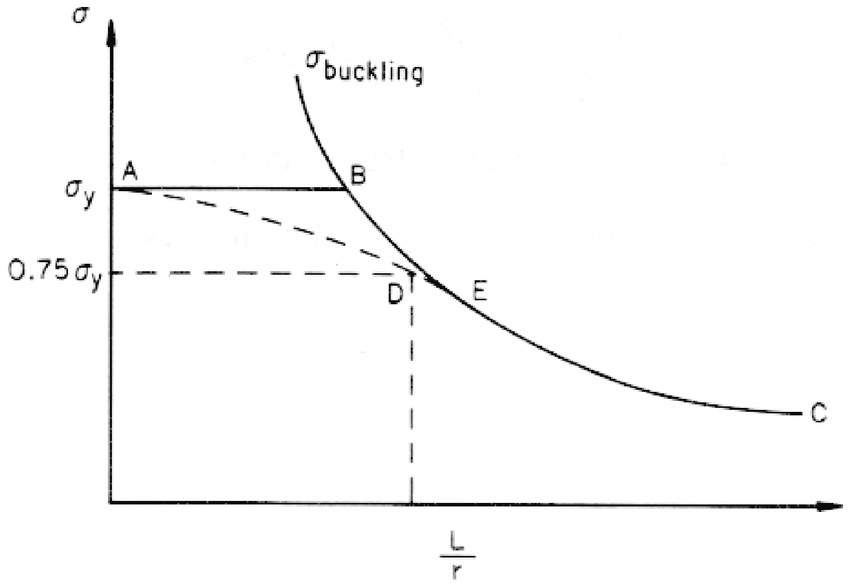


Fig. 10.8

However, due to the residual stress occurred in the steel member during the manufacturing, the actual interaction curve is represented by the curve ADC. Therefore, AISC use a factor of 0.7 to account for this effect and

$$\frac{b}{t} = 114\sqrt{\frac{K}{\sigma_y}}$$

Single Angles

Consider the leg AB of a single angle as shown in Fig. 10.9a as a plate. The plate AB has a free support at point B and has simply supported support at point A since the point A can only rotate due to the deflection. Thus, $K = 0.456 + \frac{b^2}{a^2}$ and $K_{\min} = 0.456$. Then,

$$\frac{b}{t} = 114\sqrt{\frac{0.456}{\sigma_y}} = \frac{76}{\sqrt{\sigma_y}}$$

Double-Angles

Due to the symmetry of the double angles as shown in Fig. 10.9b, the possibility of rotation of the section under the axial compression load is significantly reduced from the previous case. Thus, AISC uses the average of the case 2 (simply supported-free) and the average of the case 1 (fixed-free) and the case 2 (simply supported-free).

$$K = \frac{0.456 + \dfrac{0.456 + 1.328}{2}}{2} = 0.674$$

Therefore, we have

$$\frac{b}{t} = 114\sqrt{\frac{0.674}{\sigma_y}} = \frac{95}{\sqrt{\sigma_y}}$$

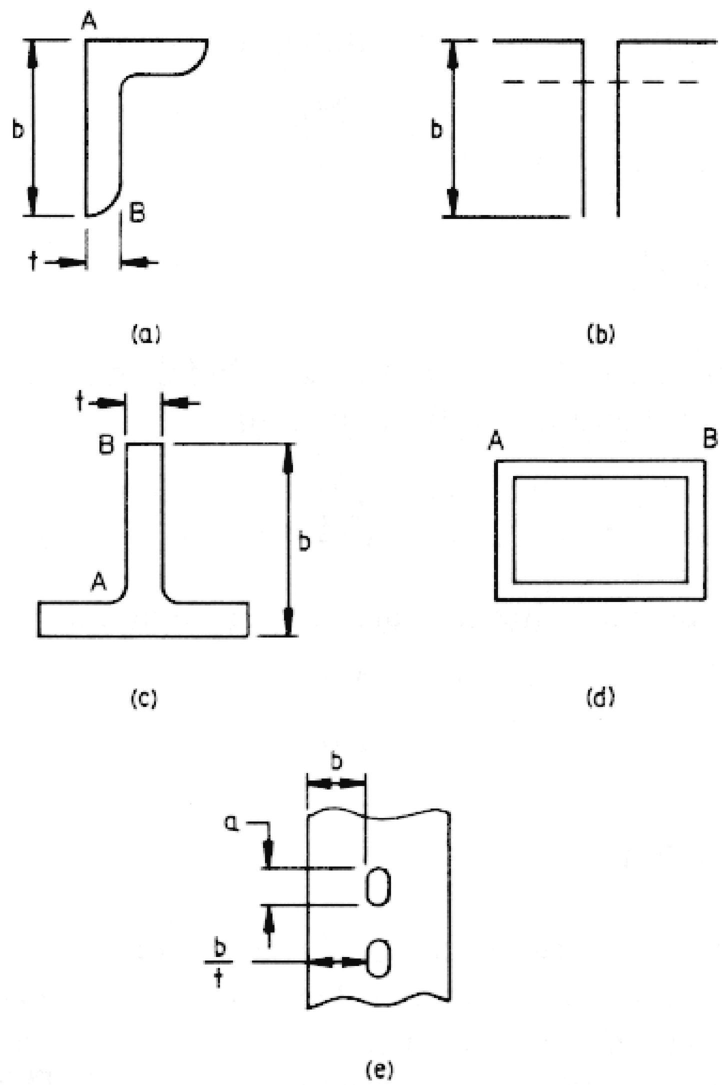


Fig. 10.9

Stems of T-section

Consider the stem of the T-section as shown in Fig. 10.9c as a plate. The support at point A is considered as a fixed support and the support at point B is considered as a free support. Therefore, the K value is 1.328 for case 1.

$$\frac{b}{t} = 114\sqrt{\frac{1.328}{\sigma_y}} = \frac{132}{\sqrt{\sigma_y}}$$

In practice, the AISC reduces the coefficient of this equation from 132 to 127.

Flanges of Box Sections

Consider the flange of the box section as shown in Fig. 10.9c as a plate. The support at point A and B can be conservatively considered as simply supported. Therefore, $K = 4.0$.

$$\frac{b}{t} = 114\sqrt{\frac{4.0}{\sigma_y}} = \frac{228}{\sqrt{\sigma_y}}$$

The AISC increases the coefficient of this equation from 228 to 238 to match the experimental results.

Perforated Cover plates

For the perforated plate as shown edge in Fig. 10.9e, the supports of the plate between the perforation and the edge are assumed to be fixed. This is because the continuous areas between the perforations add more rigidity to the plate. If the ratio of the dimension a and b of the perforated plate is equal to one, the value of K is about 7.69. This value is higher than that obtained in case 3 since it is based on the smallest possible value of K. Thus,

$$\frac{b}{t} = 114\sqrt{\frac{7.69}{\sigma_y}} = \frac{317}{\sqrt{\sigma_y}}$$

Other Compressed Members

Other compressed members are assumed to have the K values between 4.0 for simply supported edge to 7.0 for fixed edges. The AISC uses the value of $K = 4.90$.

$$\frac{b}{t} = 114\sqrt{\frac{4.90}{\sigma_y}} = \frac{253}{\sqrt{\sigma_y}}$$

References

1. **Advanced Mechanics of Materials**; 4th Edition, A.P. Boresi and O.M. Sidebottom, John Wiley & Sons, 1985
2. **Advanced Mechanics of Materials**; 2nd Edition, R.D. Cook and W.C. Young, Prentice Hall, 1999
3. **Theory of Elastic Stability**; 2nd Edition, S.P. Timoshenko and J.M. Gere, McGraw-Hill, 1963
4. **Theory of Elasticity**; 3rd Edition, S.P. Timoshenko and J.N. Goodier, McGraw-Hill, 1970
5. **Theory of Plates and Shells**; 2nd Edition, S.P. Timoshenko and S. Woinowsky-Krieger, McGraw-Hill, 1970
6. **Mechanical Behavior of Materials**; 2nd Edition, N.E. Dowling, Prentice Hall, 1999
7. **Mechanics of Materials**; 3th Edition, R.C. Hibbeler, Prentice Hall, 1997